

A comparison of smoothers for state-constrained optimal control problems

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Optimal control problems governed by partial differential equations with state constraints are considered. The state constraints are treated by two types of regularization techniques, namely the Lavrentiev type and the Moreau-Yosida type regularization. For the realization of the numerical solution, a multigrid method is applied to the regularized problems. The main purpose of this research is to compare smoothing procedures for solving state-constrained optimal control problems. Results of numerical experiments show the computational performance of both smoothing strategies.

INTRODUCTION

The objective of this paper is to contribute to the development of numerical techniques for solving partial differential equations (PDEs) with pointwise state constraints. In the recent years, ample attention has been given to the numerical solution of its unconstrained counterpart. See (Borzi et al. 2002, Lass et al. 2009, Takacs and Zulehner 2011, Vallejos and Borzi 2008) and the references given therein. Similarly, optimal control problems with control-constraints are also well-studied, take for example (Borzi and Kunisch 2005, Borzi and Schulz 2009, Engel and Griebel 2011, Hinze and Vierling 2012, Lass et al. 2009). On the contrary, less attention has been given to the numerical solution of optimal control problems with state constraints. This is due to the fact that problems of this type are in general difficult to solve numerically because of the lack of regularity of the Lagrange multiplier (Bergounioux and Kunisch 2003, Casas 1986,

Meyer et al. 2007). In order to deal with the numerical difficulties, two different regularization concepts were recently proposed. Initially, Ito and Kunisch (Ito and Kunisch 2003) introduced a regularization approach of the Moreau-Yosida type. This solves the unconstrained problem, wherein the cost functional is penalized by the existence of the state constraints. The solution to the regularized problem converges to the solution of the original problem as the regularization parameter approaches infinity (de los Reyes and Yousept 2009, Ito and Kunisch 2003). Later, Meyer, Rösch and Tröltzsch (Meyer et al. 2006) introduced the Lavrentiev type regularization which approximates the state constraints by mixed control-state type of constraints. In this case, the solution to the regularized problem converges to the solution of the original problem for regularization parameters tending to zero (Meyer et al. 2006, Meyer et al. 2007, Tröltzsch and Yousept 2009).

The main contribution of this paper is to develop a numerical method for solving optimal control problems with state constraints. The state constraints are treated by a Lavrentiev type regularization, see for example (Borzi 2008, Borzi and Andrade 2012, Meyer et al. 2006, Meyer et al. 2007, Tröltzsch and Yousept 2009, Vallejos 2012), and a Moreau-Yosida type regularization (Bergounioux et al. 2006, de los Reyes and Kunisch 2006, de los Reyes and Yousept 2009, Hintermüller and Kunisch 2006, Hintermüller and Kunisch 2009, Ito and Kunisch 2003, Meyer and Yousept 2009). To the author's best knowledge, the application of a collective smoothing multigrid (CSMG) together with a Moreau-Yosida type regularization has never been investigated. On the other hand, Lavrentiev regularized problems incorporated with CSMG is already available, see (Borzi 2008,

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Borzi and Andrade 2012, Vallejos 2012). The goal of this research is to discuss these two methods and the emphasis is on the comparison of the resulting smoothing procedures. The idea for the construction of the smoothers is to exploit the properties of the corresponding optimality systems. However, the comparison of the two regularization techniques does not involve a theoretical discussion. This would go beyond the scope of this research. For a detailed discussion on the theoretical aspects of state-constrained optimal control problems, see (Bergounioux and Kunisch 2003, Casas 1986, Meyer et al. 2007) and the references therein.

This paper is organized as follows: the Lavrentiev and Moreau-Yosida regularized problems are formulated and their corresponding optimality systems are presented. This will then be followed by a detailed description of appropriate multigrid smoothers with respect to the finite difference discretization procedure. Finally, to complete this paper, some numerical experiments were performed to demonstrate the efficiency of both algorithms.

Lavrentiev type regularization

Let us consider linear-quadratic elliptic optimal control problems with pointwise box constraints on the state. Our aim is to introduce a model problem wherein the generalization to more complex cost functionals and differential equations are possible.

In this section, the state constraints are treated by a Lavrentiev type regularization. We shall focus on the state-constrained optimal control problem formulated as follows:

$$\begin{aligned} \text{minimize} \quad & J(y, u) := \frac{1}{2} \int_{\Omega} (y - z)^2 dx + \frac{\alpha}{2} \int_{\Omega} u^2 dx \\ \text{subject to} \quad & -\Delta y - u = f \quad \text{in } \Omega \\ & y = 0 \quad \text{on } \partial\Omega \\ \text{and} \quad & -\psi \leq y \leq \psi \quad \text{a.e. in } \Omega, \end{aligned} \quad (1)$$

where Ω is a bounded domain and $\partial\Omega$ the boundary of Ω . The functional J is a quadratic cost functional of the tracking type, where $z \in L^2(\Omega)$ is the desired state, the functions f and ψ are fixed functions in $L^2(\Omega)$ and $\alpha > 0$ is the weight of the cost of the control. For the existence and uniqueness of the solution to (1), we refer the reader to (Bergounioux and Kunisch 2003, Casas 1986, Meyer et al. 2007) and the references therein.

The presence of pointwise state constraints makes the problem difficult to solve. Hence by using the Lavrentiev type regularization, the pointwise state constraints $-\psi(x) \leq y(x) \leq \psi(x)$ can be approximated by

$$-\psi(x) \leq y(x) + \varepsilon u(x) \leq \psi(x),$$

where $\varepsilon > 0$ is a fixed regularization parameter. With this approximation, (1) together with the mixed control-state constraints result to the regularized problem

$$\begin{aligned} \text{minimize} \quad & J(y, u) := \frac{1}{2} \int_{\Omega} (y - z)^2 dx + \frac{\alpha}{2} \int_{\Omega} u^2 dx \\ \text{subject to} \quad & -\Delta y - u = f \quad \text{in } \Omega \\ & y = 0 \quad \text{on } \partial\Omega \\ \text{and} \quad & -\psi \leq y + \varepsilon u \leq \psi \quad \text{a.e. in } \Omega. \end{aligned} \quad (2)$$

The solution to the regularized problem (2) converges to the solution of the original problem (1) when the regularization parameter ε converges to zero (Meyer et al. 2006, Meyer et al. 2007, Tröltzsch and Yousept 2009). This property is shown in the numerical experiments, wherein for sufficiently small ε , satisfactory numerical results are obtained.

In order to solve (2), we introduce an auxiliary variable $v = y(x) + \varepsilon u(x)$, where u can be expressed in terms of y and v , such that we have

$$\begin{aligned} \text{minimize} \quad & J(y, v) := \frac{1}{2} \int_{\Omega} (y - z)^2 dx + \frac{\alpha}{2\varepsilon^2} \int_{\Omega} (v - y)^2 dx \\ \text{subject to} \quad & -\Delta y + y/\varepsilon - v/\varepsilon = f \quad \text{in } \Omega \\ & y = 0 \quad \text{on } \partial\Omega \\ \text{and} \quad & -\psi \leq v \leq \psi \quad \text{a.e. in } \Omega. \end{aligned}$$

This system is similar to an optimal control problem having a control-constrained structure with respect to the variable v . In a convex setting, solving an optimal control problem is equivalent to solving the optimality system. For a given Lagrange multiplier p which is assumed to be a function in L^2 (Meyer et al. 2006, Meyer et al. 2007, Tröltzsch 2005), the first-order necessary optimality conditions are

$$\begin{aligned} -\Delta y + y/\varepsilon - v/\varepsilon &= f \quad \text{in } \Omega, & y &= 0 \quad \text{on } \partial\Omega \\ -\Delta p + p/\varepsilon + (1 + \beta)y - \beta v &= z \quad \text{in } \Omega, & p &= 0 \quad \text{on } \partial\Omega \\ (-p/\varepsilon + \beta(v - y), t - v) &\geq 0 \quad \text{in } \Omega, \end{aligned} \quad (3)$$

where $\beta = \alpha/\varepsilon^2$ and the inequality condition holds for all t in the admissible set $V_{ad} = \{v \in L^2(\Omega) \mid -\psi \leq v \leq \psi \text{ a.e. in } \Omega\}$. The first two equations are the state equation and adjoint equation, respectively. The inequality condition is called the optimality condition, which can be defined as a pointwise projection onto the admissible set V_{ad} .

Next we discuss the second type of regularization technique which can be utilized in solving state-constrained optimal control problems.

Moreau-Yosida type regularization

In this section, we present the Moreau-Yosida type regularization technique which can solve optimal control problems with state-constraints by introducing a penalty term. Similar to the previous section, we consider model problem (1). Notice that the pointwise state constraint $-\psi \leq y \leq \psi$ can also be written as $|y| \leq \psi$.

By using the Moreau-Yosida type regularization, the main idea is to consider a penalized optimal control problem, where a

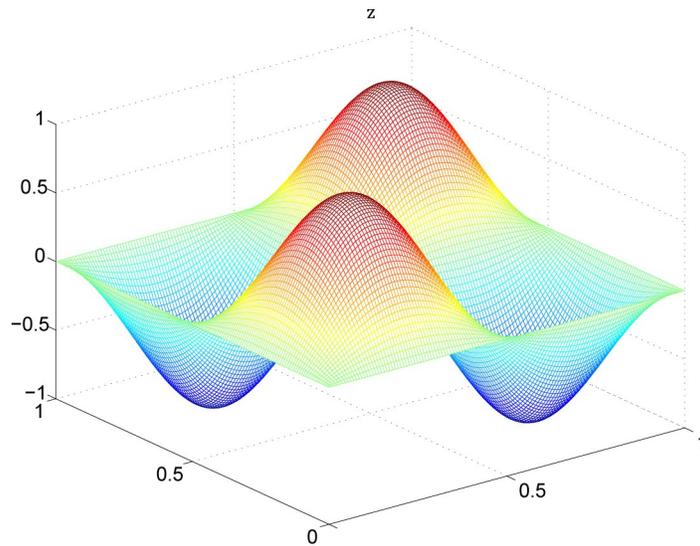


Figure 1. The target function z for Example 1.

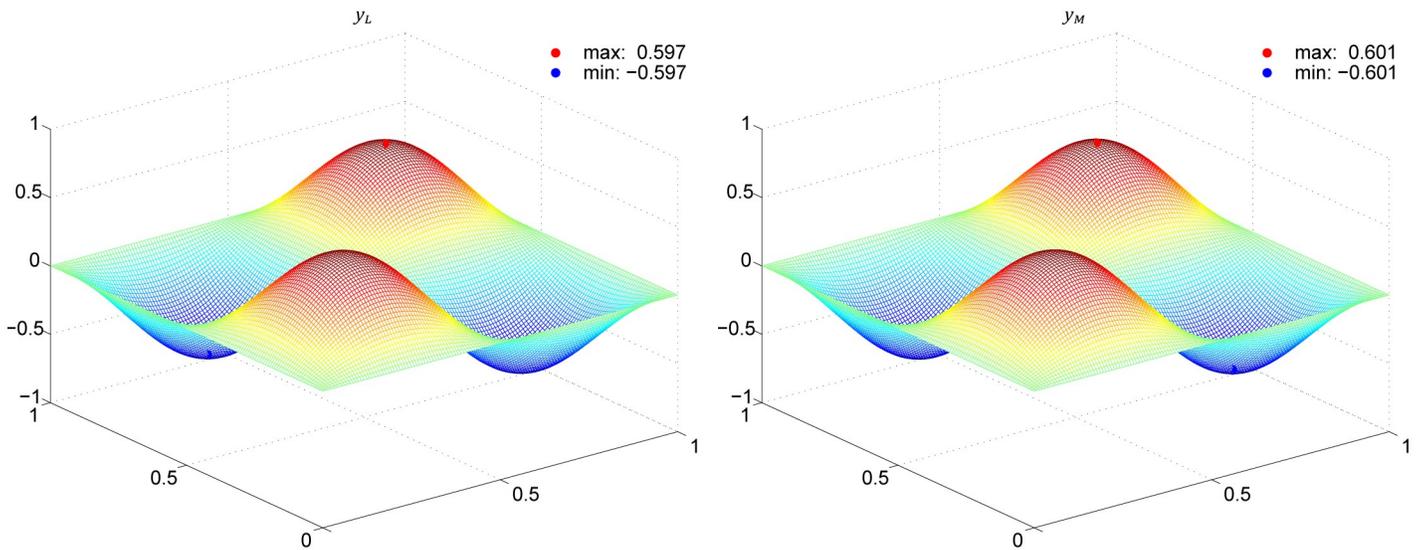


Figure 2. The Lavrentiev optimal state y_L for $\varepsilon = 10^{-4}$ (left) and Moreau-Yosida optimal state y_M for $\omega = 10^4$ (right).

Table 1. Number of Iteration and convergence results for the Lavrentiev regularized problem.

ε	N	Inter	J	$\ r(y)\ _{L^2}$	$\ r(p)\ _{L^2}$	Time(s)
10^{-2}	256×256	7	$5.5239 \cdot 10^{-2}$	$9.37 \cdot 10^{-7}$	$8.68 \cdot 10^{-9}$	1.6
	512×512	7	$5.5241 \cdot 10^{-2}$	$9.73 \cdot 10^{-7}$	$8.98 \cdot 10^{-9}$	6.4
	1024×1024	7	$5.5241 \cdot 10^{-2}$	$9.84 \cdot 10^{-7}$	$9.08 \cdot 10^{-9}$	24.7
10^{-3}	256×256	8	$4.8337 \cdot 10^{-2}$	$6.68 \cdot 10^{-6}$	$5.56 \cdot 10^{-7}$	1.9
	512×512	8	$4.8339 \cdot 10^{-2}$	$7.17 \cdot 10^{-6}$	$6.08 \cdot 10^{-7}$	7.4
	1024×1024	8	$4.8340 \cdot 10^{-2}$	$6.72 \cdot 10^{-6}$	$5.67 \cdot 10^{-7}$	28.4
10^{-4}	256×256	11	$4.8057 \cdot 10^{-2}$	$6.92 \cdot 10^{-6}$	$7.01 \cdot 10^{-6}$	2.5
	512×512	14	$4.8060 \cdot 10^{-2}$	$6.63 \cdot 10^{-6}$	$6.89 \cdot 10^{-6}$	12.8
	1024×1024	14	$4.8060 \cdot 10^{-2}$	$5.55 \cdot 10^{-6}$	$5.77 \cdot 10^{-6}$	49.4

Table 2. Number of Iteration and convergence results for the Moreau-Yosida regularized problem.

ω	N	Inter	J	$\ r(y)\ _{L^2}$	$\ r(p)\ _{L^2}$	Time(s)
10^2	256×256	8	$4.8011 \cdot 10^{-2}$	$3.36 \cdot 10^{-6}$	$4.78 \cdot 10^{-7}$	1.1
	512×512	8	$4.8013 \cdot 10^{-2}$	$3.04 \cdot 10^{-6}$	$5.30 \cdot 10^{-7}$	4.5
	1024×1024	8	$4.8014 \cdot 10^{-2}$	$2.99 \cdot 10^{-6}$	$5.54 \cdot 10^{-7}$	17.7
10^3	256×256	9	$4.8026 \cdot 10^{-2}$	$1.06 \cdot 10^{-6}$	$3.43 \cdot 10^{-7}$	1.2
	512×512	8	$4.8028 \cdot 10^{-2}$	$7.76 \cdot 10^{-6}$	$5.02 \cdot 10^{-6}$	4.5
	1024×1024	8	$4.8029 \cdot 10^{-2}$	$4.92 \cdot 10^{-6}$	$4.84 \cdot 10^{-6}$	17.2
10^4	256×256	10	$4.8034 \cdot 10^{-2}$	$6.10 \cdot 10^{-6}$	$9.99 \cdot 10^{-6}$	1.3
	512×512	8	$4.8036 \cdot 10^{-2}$	$3.69 \cdot 10^{-6}$	$4.64 \cdot 10^{-6}$	4.5
	1024×1024	8	$4.8037 \cdot 10^{-2}$	$4.26 \cdot 10^{-6}$	$5.12 \cdot 10^{-6}$	17.1

penalty term is added to the objective functional to compensate for the removal of the pointwise state constraints (de los Reyes and Yousept 2009, Ito and Kunisch 2003, Meyer and Yousept 2009). The penalized optimal control problem is of the form

$$\begin{aligned} \text{minimize} \quad & J(y, u) := \frac{1}{2} \int_{\Omega} (y - z)^2 dx + \frac{\alpha}{2} \int_{\Omega} u^2 dx + \frac{\omega}{2} \int_{\Omega} (|y| - \psi)^+ dx \\ \text{subject to} \quad & -\Delta y - u = f \quad \text{in } \Omega \\ & y = 0 \quad \text{on } \partial\Omega, \end{aligned} \quad (4)$$

where $\omega > 0$ denotes the penalization parameter and $(|y| - \psi)^+ = \max(0, |y| - \psi)$ in the pointwise almost everywhere sense. This means that the penalty term is zero if y is within the given admissible set $V_{ad} = \{y \in L^2(\Omega) \mid -\psi \leq y \leq \psi \text{ a.e. in } \Omega\}$. Otherwise, it is nonzero and hence penalizes the objective functional, i.e. we have

$$(|y| - \psi)^+ = \begin{cases} 0, & |y| \leq \psi \\ |y| - \psi, & |y| > \psi. \end{cases}$$

Problem (4) has the same structure as an unconstrained optimal control problem. This means that this approach may also be used for optimal control problems with both control and state constraints, which is not possible for the Lavrentiev case. With the introduction of the penalty term, the derivation of the optimality system is more involved in the Moreau-Yosida type regularization. The difficulty is encountered in the formulation of the adjoint equation which depends strongly on the state variable y . The state equation and the optimality condition are given as follows:

$$\begin{aligned} -\Delta y - u - f &= 0 \quad \text{in } \Omega \\ y &= 0 \quad \text{on } \partial\Omega \\ \alpha u - p &= 0 \quad \text{in } \Omega. \end{aligned}$$

On the other hand, the adjoint equation has two cases. For the first case, where $|y| \leq \psi$, we have $-\Delta p + y - z = 0$ in Ω together with $p = 0$ on $\partial\Omega$. While for the case $|y| > \psi$ there are two sub-cases

$$\begin{aligned} -\Delta p + y - z + \omega(y - \psi) &= 0 \quad \text{for } y > \psi \\ -\Delta p + y - z + \omega(y + \psi) &= 0 \quad \text{for } y < -\psi. \end{aligned}$$

The solution to the regularized problem (4) converges to the solution of the original problem (1) as the regularization parameter ω approaches infinity (de los Reyes and Yousept 2009, Ito and Kunisch 2003, Meyer and Yousept 2009).

Next, we discuss the smoothers for the Lavrentiev regularized and Moreau-Yosida regularized problems. Optimal control problems in a finite element framework are presented for example in (Engel and Griebel 2011, Lass et al. 2009, Meyer et al. 2007, Meyer and Yousept 2009, Takacs and Zulehner 2011). For a simple comparison of the resulting smoothers, we focus on the finite difference approximation.

Smoother for a Lavrentiev regularized problem

Let us start by recalling the optimality system (3) for the state constrained problem using Lavrentiev type regularization. Here we consider its corresponding discrete version. Let Ω be a rectangular domain and consider a sequence of nested discretization grids $\{\Omega_k\}$, i.e. $\Omega_1 \subset \Omega_2 \subset \dots \subset \Omega_L = \Omega$, where $k = L$ denotes the finest level. Moreover, the mesh size of each grid Ω_k is h_k , $k = 1, 2, \dots, L$. Associated to this sequence of grids is a sequence of finite difference spaces $V_1 \subset V_2 \subset \dots \subset V_L = V$. For each grid level k , the discrete problem is represented by $A_k w_k = f_k$, where A_k defines the discrete operator on Ω_k . For the Lavrentiev regularized problem $w_k := (y_k, p_k, v_k)$ while for the Moreau-Yosida case we have $w_k := (y_k, p_k, u_k)$. The multigrid scheme presented here is based on the well-known full approximation storage (FAS) scheme (Brandt 1977). The collective smoothing multigrid (CSMG) solves the optimal control problem by solving the corresponding optimality system. For solving the discrete problem $A_k w_k = f_k$, we denote the smoother by S_k in the CSMG Algorithm.

CSMG Algorithm

- 1: Initialize $w_k^{(0)}$ to be an initial approximation at resolution k .
- 2: **if** $k = 1$ **then**
- 3: solve $A_k w_k = f_k$ exactly and return.
- 4: **else**
- 5: *Pre-smoothing*. Apply γ_1 iterations of a smoothing algorithm to the problem at resolution k .
 $w_k^{(\ell)} = S_k(w_k^{(\ell-1)})$, $\ell = 1, 2, \dots, \gamma_1$
- 6: *Coarse grid problem*. Compute the fine-to-coarse residual correction
 $\tau_{k-1} = A_{k-1}(I_k^{k-1} w_k^{(\gamma_1)} - I_k^{k-1} A_k(w_k^{(\gamma_1)}))$, $f_{k-1} = I_k^{k-1} f_k + \tau_{k-1}$.
Apply γ cycles of CSMG (γ_1, γ_2) to the coarse grid problem
 $A_{k-1} w_{k-1} = f_{k-1}$ to obtain w_{k-1} .
- 7: *Coarse grid correction*.
 $w_k^{(\gamma_1+1)} = w_k^{(\gamma_1)} + I_{k-1}^k (w_{k-1} - I_k^{k-1} w_k^{(\gamma_1)})$.
- 8: *Post-smoothing*. Apply γ_2 iterations of a smoothing algorithm to the problem at resolution k .
 $w_k^{(\ell)} = S_k(w_k^{(\ell-1)})$, $\ell = \gamma_1 + 2, \dots, \gamma_1 + \gamma_2 + 1$
- 9: **end if**

Here the operators $I_k^{k-1}: V_k \rightarrow V_{k-1}$ and $I_{k-1}^k: V_{k-1} \rightarrow V_k$ are chosen to be full-weighting restriction and bilinear interpolation, respectively. We choose the most commonly used multigrid transfer operators for ease of computation. In stencil form, these operators are given by

$$I_k^{k-1} = \frac{1}{16} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix} \quad \text{and} \quad I_{k-1}^k = \frac{1}{4} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}.$$

The parameter γ is called the cycle index which characterizes the multigrid cycle being used. For $\gamma = 1$, we have a V -cycle and for $\gamma = 2$, a W -cycle.

Table 3. Number of Iteration and convergence results for Example 2.

	ω	N	Inter	J	$\ r(y)\ _{L^2}$	$\ r(p)\ _{L^2}$	Time(s)
Lavrentiev		256×256	7	$1.8856 \cdot 10^{-3}$	$7.59 \cdot 10^{-6}$	$4.66 \cdot 10^{-8}$	2.0
		512×512	7	$1.8866 \cdot 10^{-3}$	$5.14 \cdot 10^{-6}$	$3.26 \cdot 10^{-8}$	8.5
		1024×1024	7	$1.8868 \cdot 10^{-3}$	$5.43 \cdot 10^{-6}$	$3.44 \cdot 10^{-8}$	35.4
Moreau-Yosida		256×256	11	$1.8846 \cdot 10^{-3}$	$6.10 \cdot 10^{-6}$	$3.19 \cdot 10^{-7}$	2.8
		512×512	10	$1.8855 \cdot 10^{-3}$	$3.15 \cdot 10^{-6}$	$1.07 \cdot 10^{-7}$	8.8
		1024×1024	10	$1.8857 \cdot 10^{-3}$	$7.63 \cdot 10^{-6}$	$3.01 \cdot 10^{-7}$	37.4

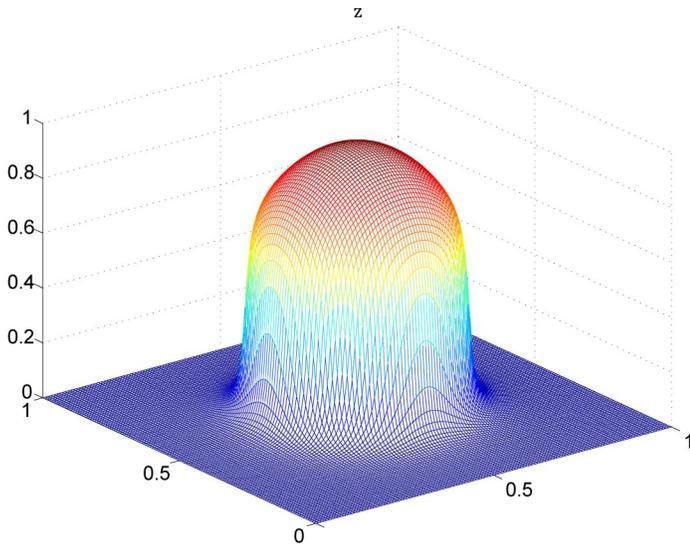


Figure 3. The target function z for Example 2.

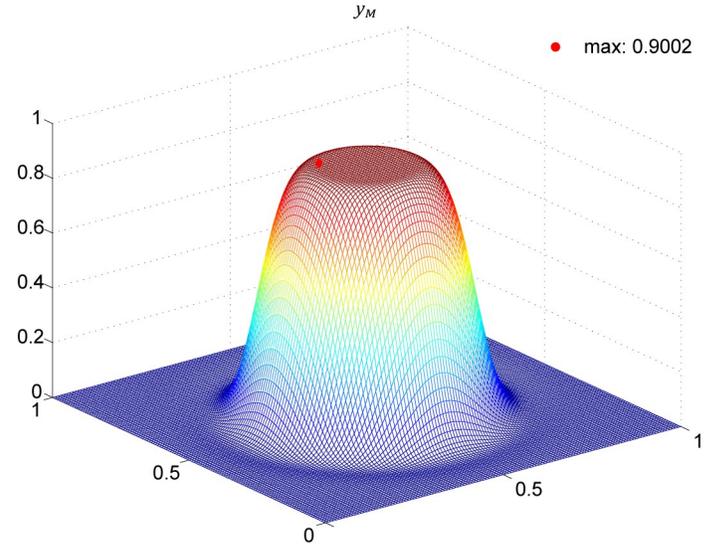
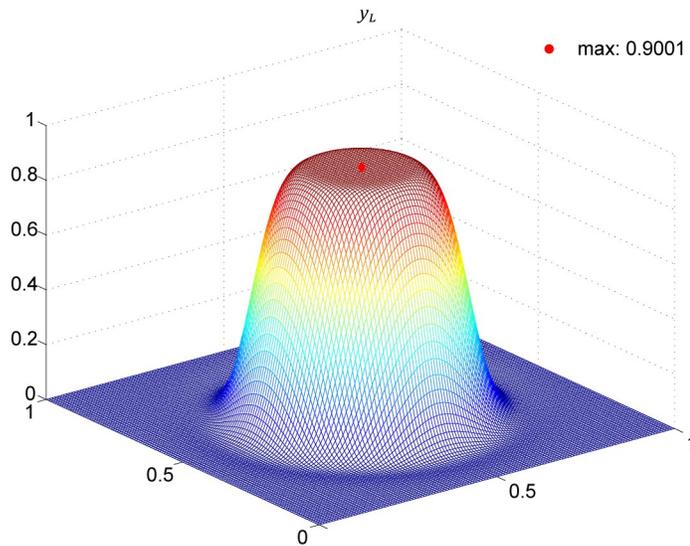


Figure 4. The Lavrentiev optimal state y_L (left) and the Moreau-Yosida optimal state y_M (right) for Example 2.

In the following discretization, Δ_k denotes the five-point stencil for the Laplace operator and we get

$$\begin{aligned} -\Delta_k y_k + y_k/\varepsilon - v_k/\varepsilon &= f_k \\ -\Delta_k p_k + p_k/\varepsilon + (1 + \beta)y_k - \beta v_k &= z_k \\ (-p_k/\varepsilon + \beta(v_k - y_k), t_k - v_k) &\geq 0. \end{aligned}$$

For a detailed derivation of a smoothing algorithm for a Lavrentiev type regularized problem, let $h = h_k$ and the grid points $x = (i_h, j_h)$ in Ω_k with indices i, j be arranged lexicographically and we define

$$\begin{aligned} A &:= -(y_{i-1,j} + y_{i+1,j} + y_{i,j-1} + y_{i,j+1}) - h^2 f_{i,j} \\ B &:= -(p_{i-1,j} + p_{i+1,j} + p_{i,j-1} + p_{i,j+1}) - h^2 z_{i,j}, \end{aligned}$$

wherein the values A and B are considered constant during the update of the variables at i, j . Then we have

$$\begin{aligned} A + (4 + h^2/\varepsilon)y_{i,j} - (h^2/\varepsilon)v_{i,j} &= 0 \\ B + (4 + h^2/\varepsilon)p_{i,j} + (1 + \beta)h^2 y_{i,j} - \beta h^2 v_{i,j} &= 0 \\ (-p_{i,j}/\varepsilon - \beta y_{i,j} + \beta v_{i,j}, t_{i,j} - v_{i,j}) &\geq 0, \end{aligned}$$

where the inequality holds for all t in the admissible set $(V_{ad})_k = \{v \in L^2_k(\Omega_k) \mid -\psi \leq v \leq \psi \text{ in } \Omega_k\}$. The update for the variables $y_{i,j}$ and $p_{i,j}$ can be computed from the discrete state and adjoint equations and can be written explicitly as

$$\begin{aligned} y_{i,j} &= \frac{h^2 v_{i,j} - A\varepsilon}{h^2 + 4\varepsilon} \\ p_{i,j} &= \frac{\varepsilon(-4B\varepsilon - h^2 B - h^4 v_{i,j} + h^2 A\varepsilon + \beta h^2 A\varepsilon + 4\beta h^2 v_{i,j} \varepsilon)}{(h^2 + 4\varepsilon)^2}. \end{aligned}$$

In this update, $y_{i,j}$ and $p_{i,j}$ are both defined as functions of $v_{i,j}$. To solve an update for $v_{i,j}$, replace $y_{i,j}$ and $p_{i,j}$ in the inequality condition. In the absence of constraints, we define an auxiliary variable $\tilde{v}_{i,j}$ to be the solution of $-p_{i,j}/\varepsilon - \beta y_{i,j} + \beta \tilde{v}_{i,j} = 0$ and we get

$$\tilde{v}_{i,j} = \frac{-4B\varepsilon - h^2B + h^2A\varepsilon - 4A\beta\varepsilon^2}{h^4 + 16\beta\varepsilon^2}.$$

Since the update for $v_{i,j}$ must be within the admissible set, then $v_{i,j}$ is obtained by projecting $\tilde{v}_{i,j}$ onto $(V_{ad})_k$ such that

$$v_{i,j} = \begin{cases} -\psi_{i,j}, & \text{if } \tilde{v}_{i,j} < -\psi_{i,j} \\ \tilde{v}_{i,j}, & \text{if } -\psi_{i,j} \leq \tilde{v}_{i,j} \leq \psi_{i,j} \\ \psi_{i,j}, & \text{if } \tilde{v}_{i,j} > \psi_{i,j} \end{cases}$$

holds. The update for the variables $y_{i,j}$, $p_{i,j}$ and $v_{i,j}$ completes the smoothing algorithm for the state constrained optimal control problem with Lavrentiev type regularization.

Smoother for a Moreau-Yosida regularized problem

For the state constrained problem using Moreau-Yosida type regularization, the resulting optimality system results to three different cases. We use the same notations for the discretization of the optimality system as in the Lavrentiev type. Since using a Moreau-Yosida type regularization results to an unconstrained optimal control problem, the formulation of the update for the variables y , p and u is straightforward. One needs to solve for y and p from the state and the adjoint equations, respectively. Then by plugging these to the optimality condition, u is obtained. For the case $|y| \leq \psi$, the discrete version of the optimality system gives

$$\begin{aligned} A + 4y_{i,j} - h^2u_{i,j} &= 0 \\ B + 4p_{i,j} + h^2y_{i,j} &= 0 \\ \alpha u_{i,j} - p_{i,j} &= 0, \end{aligned}$$

where the update for $y_{i,j}$, $p_{i,j}$ and $u_{i,j}$ are as follows:

$$y_{i,j} = \frac{h^2u_{i,j} - A}{4}, \quad p_{i,j} = \frac{-4B - h^4u_{i,j} + h^2A}{16}, \quad u_{i,j} = \frac{-4B + h^2A}{16\alpha + h^4}.$$

For the second case with $|y| > \psi$, have two subcases, namely $y > \psi$ and $y < -\psi$. For the first subcase, we have

$$\begin{aligned} A + 4y_{i,j} - h^2u_{i,j} &= 0 \\ B + 4p_{i,j} + h^2y_{i,j} + \omega h^2(y_{i,j} - \psi) &= 0 \\ \alpha u_{i,j} - p_{i,j} &= 0, \end{aligned}$$

with the corresponding update

$$\begin{aligned} y_{i,j} &= \frac{h^2u_{i,j} - A}{4} \\ p_{i,j} &= \frac{-4B - h^4u_{i,j} + h^2A - \omega h^4u_{i,j} + \omega h^2A + 4\omega h^2\psi}{16} \\ u_{i,j} &= \frac{-4B + h^2A + \omega h^2A + 4\omega h^2\psi}{16\alpha + h^4 + \omega h^4}. \end{aligned}$$

And finally for $y < -\psi$ we have

$$\begin{aligned} A + 4y_{i,j} - h^2u_{i,j} &= 0 \\ B + 4p_{i,j} + h^2y_{i,j} + \omega h^2(y_{i,j} + \psi) &= 0 \\ \alpha u_{i,j} - p_{i,j} &= 0, \end{aligned}$$

which results to the following update

$$\begin{aligned} y_{i,j} &= \frac{h^2u_{i,j} - A}{4} \\ p_{i,j} &= \frac{-4B - h^4u_{i,j} + h^2A - \omega h^4u_{i,j} + \omega h^2A + 4\omega h^2\psi}{16} \\ u_{i,j} &= \frac{-4B + h^2A + \omega h^2A - 4\omega h^2\psi}{16\alpha + h^4 + \omega h^4}. \end{aligned}$$

The update for the variables $y_{i,j}$, $p_{i,j}$ and $u_{i,j}$ generates the smoothing algorithm for the state constrained optimal control problem with Moreau-Yosida type regularization. With a careful implementation of the different cases resulting from the optimality system of the penalized problem, a robust multigrid strategy is guaranteed for sufficiently large penalization parameter ω .

Numerical results

In the previous sections, we present the proposed multigrid method for solving the state-constrained optimal control problem (1). We now report on the results of the numerical experiments to compare the computational performance of the two different smoothing procedures. All numerical computations were performed on a PC with 2.2 GHz processor. For all the numerical examples, we consider the optimal control problem (1) on a unit square domain $\Omega = (0,1)^2 \subset \mathbb{R}^2$. We use the V -cycle ($\gamma = 1$) with 2 pre- and post-smoothing steps and run all the examples on a uniform square mesh with 1024×1024 interior grid points on the finest mesh. The number of iterations and the CPU time in seconds are reported together with the L^2 -norm of the state and adjoint equation residuals, $\|r(y)\|_{L^2}$ and $\|r(p)\|_{L^2}$, respectively. The algorithm terminates when the tolerance $\text{tol} = 10^{-6}$ for the norm of the residuals is reached. The cost functional J is also reported, wherein for ease of comparison J stands for the original cost and not on the one resulting from the regularized problems.

Example 1. Consider problem (1) with $f = 0$ and z given by $z(x_1, x_2) = \sin(2\pi x_1)\sin(2\pi x_2)$. We also consider pointwise state constraints $-0.6 \leq y \leq 0.6$ and all unknown variables are initialized to zero. The target function z is shown in Figure 1. For small values of the weighing parameter α in the cost functional J , we expect an accurate tracking, i.e. $\|y - z\|_{L^2}$ is sufficiently small. This implies that the state solution y inherits the characteristics of the target function z and at the same time taking into consideration the given state constraints. In all the calculations $\alpha = 10^{-4}$ and we consider the regularization parameters $\varepsilon = \{10^{-2}, 10^{-3}, 10^{-4}\}$ for the Lavrentiev type and $\omega = \{10^2, 10^3, 10^4\}$ for the Moreau-Yosida type smoothing.

First, let us look at the Lavrentiev regularized problem. Nu-

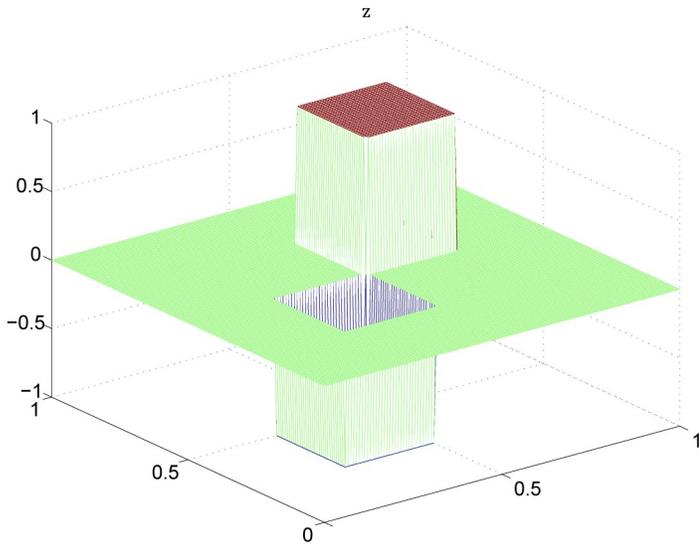


Figure 5. The target function z for Example 3.

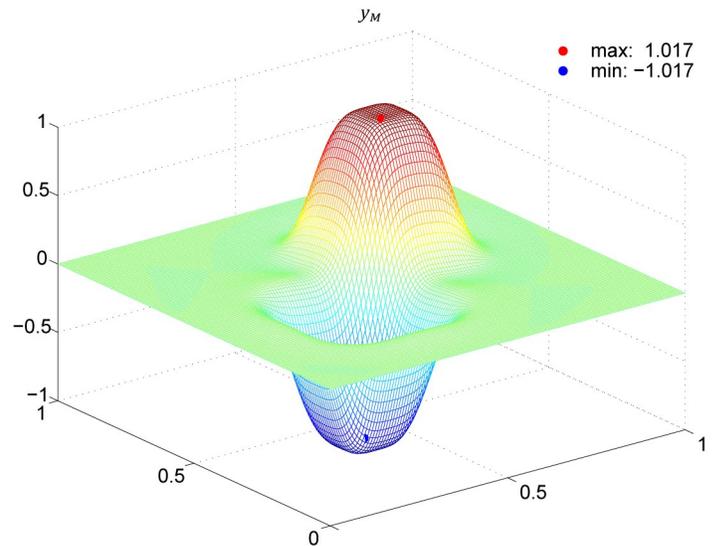
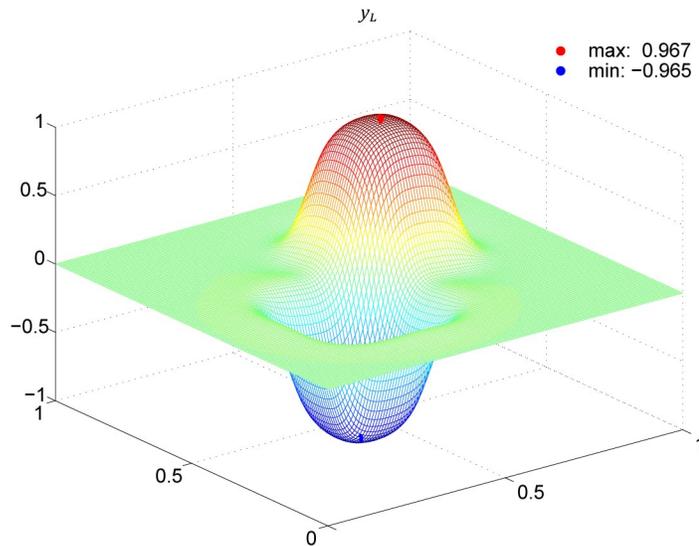


Figure 4. The Lavrentiev optimal state y_L (left) and the Moreau-Yosida optimal state y_M (right) for Example 3.

numerical results are shown in Table 1, which show that the number of iterations is almost independent of the mesh size but not of the regularization parameter ε . For smaller values of ε , better approximations can be obtained. However, the problem becomes harder to solve which is expressed by the increase in the number of iterations. For each regularization parameter, the CPU time increases by a factor of four when halving the mesh size. This shows an optimal computational complexity of the multigrid approach. The numerical solution for the state variable y is shown in Figure 2 (left).

Next, we consider the Moreau-Yosida regularized problem. Numerical results are reported in Table 2. The results show that the number of iterations is almost independent of the regularization parameter ω and of the mesh size. For larger values of ω ,

Table 4. Number of Iteration and convergence results for Example 3.

	ω	N	Inter	J	$\ r(y)\ _{L^2}$	$\ r(p)\ _{L^2}$	Time(s)
Lavrentiev		256×256	9	$1.1478 \cdot 10^{-2}$	$9.37 \cdot 10^{-6}$	$6.30 \cdot 10^{-8}$	2.3
		512×512	9	$1.1389 \cdot 10^{-2}$	$6.86 \cdot 10^{-6}$	$5.71 \cdot 10^{-8}$	10.8
		1024×1024	10	$1.1339 \cdot 10^{-2}$	$3.41 \cdot 10^{-6}$	$3.02 \cdot 10^{-8}$	50.3
Moreau-Yosida		256×256	9	$1.1425 \cdot 10^{-2}$	$3.51 \cdot 10^{-6}$	$2.16 \cdot 10^{-8}$	1.8
		512×512	9	$1.1335 \cdot 10^{-2}$	$2.57 \cdot 10^{-6}$	$1.61 \cdot 10^{-8}$	7.9
		1024×1024	8	$1.1285 \cdot 10^{-2}$	$6.44 \cdot 10^{-6}$	$3.78 \cdot 10^{-8}$	30.0

better approximations can be achieved. Notice that the problem exhibits a minimal difference in the number of iterations and CPU time for different values of the regularization parameter ω . As in the Lavrentiev case, the CPU time increases by a factor of four when halving the mesh size. These results suggest that the Moreau-Yosida approach demonstrates the characteristics of a multigrid algorithm of not only being mesh independent but also parameter independent with respect to the number of iterations. The numerical solution for the state variable y is shown in Figure 2 (right).

Example 2. In this example, we use the same settings as Example 1, with $f=1$ and the target function z given by

$$z = 16x_1x_2(1-x_1)(1-x_2)\left(0.5 + \frac{1}{\pi} \tan^{-1}(200(0.0625 - (x_1 - 0.5)^2 - (x_2 - 0.5)^2))\right),$$

see Figure 3. The given target function has a circular inner layer located at the circle $(x_1 - 0.5)^2 - (x_2 - 0.5)^2 = 0.25^2$. This layer is characterized by the sudden transition in the numerical solution which may cause oscillations for a coarser mesh. In this case, we impose only an upper bound to the state variable, i.e. $0 \leq y \leq 0.9$. The numerical results are shown in Table 3. For the Lavrentiev case, we utilize $\varepsilon = 10^{-4}$ and for the Moreau-Yosida $\omega = 10^3$. These regularization parameters are chosen in such a way that one can easily compare the corresponding cost functional J . Similar to Example 1, the number of iteration is also mesh independent and the computational time are almost the same for both methods. Figure 4 exhibits the Lavrentiev optimal state y_L (left) and the Moreau-Yosida optimal state y_M (right). The weighing parameter α in the cost functional J is $\alpha = 10^{-6}$. The choice of a small weighing parameter is to show how well the state solution approximates the desired target function. We can see in Figure 4 that the state solution y inherits the characteristics of the target function z and at the same time taking into consideration the given state constraints.

Example 3. Finally, let us consider a discontinuous target function z given by

$$z = \begin{cases} -1, & \text{if } 0.25 \leq x_1 \leq 0.5, 0.25 \leq x_2 \leq 0.5 \\ 1, & \text{if } 0.5 \leq x_1 \leq 0.75, 0.5 \leq x_2 \leq 0.75 \\ 0, & \text{otherwise} \end{cases}$$

together with $f = 1$. In this example, the target function is given in Figure 5 and the corresponding Lavrentiev optimal state y_L (left) and the Moreau-Yosida optimal state y_M (right) are shown in Figure 6. We impose the following constraints on the state variable: $-1.0 \leq y \leq 1.0$. The weighing parameter α is set to 10^{-6} and Figure 6 shows that the state solution y has the same structure as the target function z but still satisfies the state constraints. The numerical results are shown in Table 4. For the Lavrentiev case, we use $\varepsilon = 10^{-4}$ and for the Moreau-Yosida $\omega = 10^2$. The choice of these regularization parameters is for ease of comparison in the cost functional J . Similar to Example 2, the number of iteration is also mesh independent but the computational time shows that the Moreau-Yosida strategy solves the problem faster than the Lavrentiev.

By comparing the values of the cost functional J in Tables 1-4, we observe that for each corresponding regularization parameter, lower values are achieved for the Moreau-Yosida approach compared to the Lavrentiev method. There is only a minor difference in the performance, wherein the Moreau-Yosida type is less sensitive to the regularization parameter in comparison to the Lavrentiev type. This means that with Moreau-Yosida, the parameter ω does not have to be sufficiently large in order to obtain a good numerical approximation. Although both approaches give satisfactory solutions, they exhibit different characteristics. The Moreau-Yosida regularized problem has the structure of an unconstrained optimal control problem which is in general easier to solve numerically. Whereas, on the other hand, the Lavrentiev regularized problem exploits the control-

constraint framework. The advantage of a Moreau-Yosida approach is that optimal control problems with both control and state constraints can be investigated. This is in contrast to a Lavrentiev approach which is transformed into a mixed control-state problem and hence cannot incorporate an additional control constraint. Since the numerical method presented in this paper is based on the collective smoothing multigrid technique, the optimality system plays a very important role in the implementation procedure, wherein Moreau-Yosida is more rigorous in comparison to Lavrentiev. However, for appropriate values of the parameters ε and ω , the regularized problems approximate the original problem satisfactorily. Hence both the numerical methods exhibit similar computational performance and efficiency.

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CONFLICTS OF INTEREST

There are no conflicts of interest arising from this research.

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