Non-entire Solutions of the Generalized Dhombres and Related Equations

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The Generalized Dhombres Equation with analytic data has been shown to have local analytic solutions and was thought to have entire solutions if the data is entire. We show that this is not generally true. We also present non-entire solutions for several other equations related to the Generalized Dhombres Equation.

KEYWORDS

functional equation, non-entire solution, non-existence

INTRODUCTION

The generalized Dhombres equation is given by

$$f\big(zf(z)\big) = \omega\big(f(z)\big), \qquad (1)$$

where $\omega(z)$ is a holomorphic function with expansion $\omega(z) = z^{k+1} + c_{k+2}z^{k+2} + \cdots$ and $f(z)$ is the unknown function of the form $f(z) = a_k z^k + a_{k+1}z^{k+1} + \cdots$. This is a generalization of the Dhombres equation $\big(xf(x)\big) = f(x)^2$, which was introduced by Jean Dhombres in 1979 to describe a population model (Tomaschek 2011). Dhombres considered the original equation in the real domain, while Reich et al. considered in 2005 the generalized equation in the complex domain.

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The solutions to the algebraic equation (1) are described in the following theorem (Reich et al. 2005):

Theorem 1.1. *Assume that f is an analytic solution of (1) in a disc $G = \{z; |z| < \delta\}$, for some $\delta > 0$, with $f(0) = 0$, and that ω is analytic at 0. Then ω and f are of the form $\omega(y) = y^{k+1} + d_{k+2}y^{k+2} + \cdots$, $|y| < \epsilon$, for some $\epsilon > 0$ and $k \geq 1$, and $f(z) = c_k z^k + c_{k+1}z^{k+1} + \cdots$, with $c_k \neq 0$.*

Conversely, if $\omega(y) = y^{k+1} + d_{k+2}y^{k+2} + \cdots$, $|y| < \epsilon$, for some $\epsilon > 0$ and $k \geq 1$, then there is exactly one local analytic function $\tilde{f}$ with $\tilde{f}(z) = z + b_2 z^2 + \cdots$, such that the set of local analytic solutions $f(z) = c_k z^k + c_{k+1}z^{k+1} + \cdots$ of (1) is the set

$$\{f; f(z) \coloneqq \tilde{f}(c_k z^k), \text{ for some } c_k \in \mathbb{C}\} \qquad (2)$$

for $|z| < \delta$, where δ depends in general on f.

Reich (2007) claimed without proof that the following theorem will follow from the theorem above.

Theorem 1.2. *Assume that (1), where ω is an entire function, has a nonconstant holomorphic solution f with $f(0) = 0$. Then $\omega(y) = y^{k+1} + d_{k+2}y^{k+2} + \cdots$, $y \in \mathbb{C}$, for some $k \geq 1$, and $f(z) = c_k z^k + c_{k+1}z^{k+1} + \cdots$, with $c_k \neq 0$ and $z \in \mathbb{C}$. Moreover, there is an entire function $\tilde{f}$ such that the set of holomorphic solutions of (1) in $\mathbb{C}$ is the set (2) with $z \in \mathbb{C}$.*

While Theorem 1.1 completely describes the set of local solutions of (1), we could not extend its proof to work for the entire case.

In proving Theorem 1.1, Reich applied some transformations on (1) and ended up with

$$x^k U(x) = U(\psi(x)), \tag{3}$$

where $U(x) = T^{-1}(x)$ such that $T(x)$ is one of the solutions of $T(x)^k = f(x)$ and $\psi(x)$ is the solution of the equation $\psi(x)^k = \omega(x^k)$.

This last equation is a type of Generalized Böttcher equation. Let $a \neq 0$ be a complex number and $d \geq 2$ be an integer. Let $p(z) = az^d + a_{d+1}z^{d+1} + \cdots$ be holomorphic in a neighbourhood of $z = 0$. Let $P(X, Y)$ be a convergent power series with order $d + 1$. The generalized Böttcher equation is given by

$$\varphi(p(z)) = \sum_{k+l=d} a_{kl} z^k \varphi(z)^l + P(z, \varphi(z)), \tag{4}$$

with $a_{kl} \in \mathbb{C}$.

Obtaining the formal solution of (4) directly from the equation is difficult, thus the need to consider an equation equivalent to it. If we let $z = S(\zeta) = \zeta + s_2\zeta^2 + \cdots$ with S being the unique solution of $S(a\zeta^d) = p(S(\zeta))$ (see Reich 2004), then we can transform (4) to

$$\psi(a\zeta^d) = \sum_{k+l=d} a_{kl} \zeta^k \psi(\zeta)^l + Q(\zeta, \psi(\zeta)), \tag{5}$$

where $\psi(\zeta) = (\varphi \circ S)(\zeta)$ is the unknown function, a and a_{kl} have the same definitions as in (4) and $Q(X, Y)$ is a convergent power series with order at least $d + 1$.

To prove the convergence of the formal solution of (5), Reich applied the Implicit Function Theorem to a majorant equation. In view of this, we suspect that his method cannot easily be extended to cover the case with entire data.

In this paper, we will present a particular generalized Dhombres Equation with an entire $\omega(z)$ but having a non-entire solution $f(z)$. In relation to this, we will also present non-entire solutions of some generalized Böttcher and related equations.

MAIN RESULTS

Before we present the non-entire solutions of the equations mentioned earlier, we first state a useful fact concerning power series.

Lemma 2.1. *Let $f(x) = \sum a_n x^n$, $|x| < \delta$, for some $\delta > 0$. If $|a_n| \geq 1$ for infinitely many $n \in \mathbb{N}$, then $f(x)$ is not entire.*

Proof. If $|a_n| \geq 1$ for infinitely many n, then $\limsup |a_n|^{1/n} \geq 1$. Therefore, by the Cauchy-Hadamard Formula, the radius of convergence of $\sum a_n x^n$ is at most 1.

We now present a particular example of Equations (4) and (5) with entire data but with non-entire solutions.

Proposition 2.2. *Let $d \in \mathbb{N}$, $d \geq 2$, and $p(X)$ be an entire function with expansion $p(X) = p_{d+1}X^{d+1} + p_{d+2}X^{d+2} + \cdots$, where $p_i \geq 0$ for all $i \geq d + 1$. Then the equation*

$$\psi(\zeta^d) = \zeta^{d-1}\psi(\zeta) - \zeta^d\psi(\zeta) - p(\zeta) \tag{6}$$

has a solution that is not entire.

Note that (6) is a particular form of Equations (4) and (5) with $P(\zeta, \psi) = Q(\zeta, \psi) = -\zeta^d\psi - p(\zeta)$.

Proof. As in Reich (2004), let $\psi(\zeta) = \sum_{k\geq 1} a_k \zeta^k$. Substituting this expansion to (6) and using the expansion of $p(X)$, we have

$$\sum_{k=1}^{\infty} a_k \zeta^{kd} = \sum_{k=1}^{\infty} a_k \zeta^{k+d-1} - \sum_{k=1}^{\infty} a_k \zeta^{k+d} - \sum_{k=d+1}^{\infty} p_k \zeta^k. \tag{7}$$

Comparing the coefficients of ζ^d in both sides of the equation, we get $a_1 = a_1$. Thus we can choose the value of a_1 freely, and we take $a_1 = 1$. Comparing the coefficients of ζ^n, where $n = kd$ for some $k \in \mathbb{N}$, we get $a_{d(k-1)+1} = a_k + a_{d(k-1)} + p_{kd}$.

Looking at the coefficients of ζ^n in both sides of (7), where $n = kd + l$, with $0 < l \leq d - 1$, we have $a_{d(k-1)+l+1} = a_{d(k-1)+l} + p_{kd+l}$.

Hence, the coefficient a_n is given by

$$a_n = \begin{cases} a_{m+1} + a_{n-1} + p_{n+d-1}, & \text{if } n = md + 1 \text{ for some positive integer } m, \\ a_{n-1} + p_{n+d-1}, & \text{otherwise.} \end{cases}$$

To show that this solution ψ is not entire, it is enough to show that $a_n \geq 1$ for all $n \in \mathbb{N}$. We use induction to show this. Note that $a_1 = 1 \geq 1$ so suppose $a_k \geq 1$ for $k = 1,2,3, \ldots, n$. If $n = ld$, for some positive integer l, then $a_{n+1} = a_{l+1} + a_n + p_{n+d} \geq 1$. On the other hand, if $n \neq ld$ for any positive integer l, then $a_{n+1} = a_n + p_{n+d} \geq 1$. It follows that $|a_n| \geq 1$ for all n, therefore, by Lemma 2.1, the solution is not entire.

Next, we show that a special case of (3) has a non-entire solution if $\psi(x)$ is entire.

Proposition 2.3. *The equation*

$$xU(x) = U(x^2 + x^j), \tag{8}$$

where $j \in \mathbb{N}$ and $j \geq 4$, has a non-entire solution.

Note that Equation (8) is a specific case of Equation (3) with $k = 1$ and $\psi(x) = x^2 + x^j$. We can also view Equation (8) as a special case of (4).

Proof. As in Reich et al. (2005), let $U(x) = \sum_{i\geq 1} a_i x^i$. Then (8) becomes

$$\sum_{i=1}^{\infty} a_i x^{i+1} = \sum_{i=1}^{\infty} a_i (x^2 + x^j)^i. \tag{9}$$

Comparing the coefficients of x^2, we get $a_1 = a_1$ and we again set $a_1 = 1$. Comparing the coefficients of x^3, we get $a_2 = 0$. In fact, $a_i = 0$ for $i = 2, \ldots, j - 2$. If we compare the coefficients of x^j, then we obtain $a_{j-1} = a_1 = 1$.

By applying the Binomial Theorem on (9) and comparing the coefficients of x^{n+1}, we see that

$$a_n = \sum_{i=1}^{B_n} a_i S_i, \tag{10}$$

where $B_n = [\![(n + 1)/2]\!]$ and $S_i = \sum_{2i+j(i-k)=n+1} \binom{i}{k}$. It is understood that $S_i = 0$ if no integer $k \leq i$ satisfies $2i + j(i - k) = n + 1$. Otherwise, $S_i > 0$. It follows that $a_n \geq 0$ for all $n \in \mathbb{N}$ since $S_n \geq 0$ for all $n \in \mathbb{N}$.

Now let $m_n = 2^n(j - 1) - \sum_{k=0}^{n-1} 2^k = 2^n(j - 2) + 1$. We claim that $a_{m_n} \geq 1$ for all $n \in \mathbb{N}$.

Note that $m_1 = 2(j - 1) - 1$ and so,

$$a_{m_1} = \sum_{i=1}^{B_{m_1}} a_i \sum_{2k+j(i-k)=2(j-1)} \binom{i}{k}.$$

In particular, when $i = k = j - 1$, the addend is simple equal to $a_{j-1} = a_1 = 1$. Since the addends are all nonnegative, we conclude that $a_{m_1} \geq 1$.

Now suppose that $a_{m_k} \geq 1$ for $k = 1,2,\ldots,n-1$. We wish to show that $a_{m_n} \geq 1$. Note that

$$\frac{m_n+1}{2} = \frac{1}{2}\left(2^n(j-1) - \sum_{k=1}^{n-1} 2^k\right) = 2^{n-1}(j-1) - \sum_{k=1}^{n-1} 2^{k-1},$$

which is simply equal to m_{n-1}, and so

$$a_{m_n} = \sum_{i=1}^{B_{m_n}} a_i \sum_{2k+j(i-k)=m_n+1} \binom{i}{k}.$$

As before, when $i = k = (m_n+1)/2$, the addend becomes $a_{m_{n-1}}$, which is at least 1 by hypothesis.

Since $a_n \geq 1$ for infinitely many $n \in \mathbb{N}$, by Lemma 2.1, the constructed $U(x)$ is not entire.

Finally, we consider a special case of the Generalized Dhombres Equation (1). We take $\omega(z) = z^2 - z^3$.

Proposition 2.4. *The equation*

$$f\big(zf(z)\big) = [f(z)]^2 - [f(z)]^3, \qquad (11)$$

has a non-entire solution.

Proof. Let $f(z) = \sum_{n\geq 1} b_n z^n$. Substituting this to (11) and comparing the coefficients of z^2, we get $b_1^2 = b_1^2$, and so we choose $b_1 = 1$. Comparing the coefficients of z^3 and z^4, we will get $b_2 = 1$ and $b_3 = 3$, respectively. In general, comparing the coefficients of z^{n+1}, for $n > 1$, we get

$$b_n = \sum_{i=2}^{d_n} b_i \sum_{n_1+\cdots+n_i+i=n+1} b_{n_1} b_{n_2} \cdots b_{n_i} - \sum_{j=2}^{n-1} b_j b_{n+1-j} + \sum_{i+j+k=n+1} b_i b_j b_k,$$

where $d_n = [\![(n+1)/2]\!]$. Since $b_1 = 1 \in \mathbb{Z}$, this formula assures us that each b_n is an integer.

We claim that there are infinitely many nonzero coefficients b_n. We will prove this claim by contradiction. Suppose otherwise, that is, for some $m \in \mathbb{N}$, $f(z) = \sum_{n=1}^{m} b_n z^n$, where $b_m \neq 0$. Then (11) will now be

$$\sum_{n=1}^{m} b_n \left(\sum_{i=1}^{m} b_i z^{i+1}\right)^n = \left(\sum_{n=1}^{m} b_n z^n\right)^2 - \left(\sum_{n=1}^{m} b_n z^n\right)^3. \qquad (13)$$

Note that since $b_3 = 3 \neq 0$, it follows that $m \geq 3$. Looking at the degree of the polynomials on both sides of (13), we see that the polynomial on the left-hand side has degree $m(m+1)$ while the polynomial on the right-hand side has degree $3m$. Hence, we conclude that $m = 2$, which contradicts the fact that $m \geq 3$. Thus, there are infinitely many nonzero b_n. Since $b_n \in \mathbb{Z}$ for all $n \in \mathbb{N}$, there are infinitely many $n \in \mathbb{N}$ such that $|b_n| \geq 1$, and therefore by Lemma 2.1, $f(z)$ is not entire.

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