Multilevel Monte Carlo method for pricing Asian options under the square root process

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ABSTRACT

Asian options are path dependent options whose payoff depends on the average price of an asset. The price of these options does not have closed form solutions. The multilevel Monte Carlo is a method for discretization with the application of the standard Monte Carlo method. This method is easy to implement and has less computational cost than the standard Monte Carlo method. The application of this method in pricing Asian options whose underlying asset follows the geometric Brownian motion has already been studied. However, this has not been applied in pricing Asian options where the underlying asset follows the square root process. We use the Riemann scheme to discretize the temporal integral of the square root process and a simulation algorithm based on the transition density of the square root process to determine the values of the square root process at specified times. We determined that the rate of mean square convergence of the Riemann sum in approximating the temporal integral of the square root is at least of order 1. The results are compared with those obtained using standard Monte Carlo in terms of computational cost. It is shown that the use of this method with the Riemann scheme as discretization scheme is effective in terms of reducing computational cost as compared to the standard Monte Carlo method.

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INTRODUCTION

Options are usually defined as contracts giving the holder the right, not an obligation, to buy or sell the underlying asset by a certain date at a certain time. Examples of options having these standard well-defined properties are European and American call and put options. However, using only these standard options and securities make investment strategies difficult or expensive. Exotic options provide a solution to this by giving specific tailoring to risks. Basic examples of exotic options are Asian options (Mcdonald 2006).

Asian options are path dependent options where the payoff depends on the average price of the underlying asset. Hence, Asian options have lower volatility and are cheaper than European options. Asian options are useful when one is concerned about the average rate over time, when manipulation of prices can occur, and when price swings are frequent (Mcdonald 2006). The focus of our study is on pricing continuous arithmetic Asian options. Continuous arithmetic Asian options make use of the continuous arithmetic mean of the price of the underlying asset. Continuous arithmetic mean is computed by dividing the integral of the price over an interval by the length of the interval. These options are hard to price both analytically and numerically, and their price does not have a closed form solution (Boyle and Potapchik 2008).

Giles (2008) developed the multilevel Monte Carlo method. The multilevel Monte Carlo makes use of a range of different discretization levels which results in reducing the computational cost. The order of convergence of the discretization scheme plays a crucial role in reducing the computational cost. Several studies have shown the application of the multilevel Monte Carlo method in pricing various options where the underlying stock price follows the geometric Brownian motion (Giles 2008). Alaya and Kebaier (2014) used the multilevel Monte Carlo method specifically for pricing Asian options under geometric Brownian motion. They used the Riemann and trapezoidal schemes to discretize or approximate the temporal integral of the geometric Brownian motion and were able to prove limit theorems. Lapeyre and Temam (2000) also proposed time schemes approximation for the integral of geometric Brownian motion such as the Riemann and trapezoidal schemes and have shown that the use of the multilevel Monte Carlo method in this context is very efficient because the resulting Riemann and trapezoidal schemes are second order schemes.

However, the assumption of constant volatility of the geometric Brownian motion does not match market observations. The square root process is an alternative to geometric Brownian motion as a stock price model, and can be preferable to geometric Brownian motion in many circumstances (Cox and Ross 1976). Under the square root process, the instantaneous variance of the price change is inversely proportional to the stock price. This is a more realistic assumption since a fall in the stock price increases the variance of the stock (Lo, Hui and Yuen 2001). Unfortunately, this process has no closed form solution. Several researchers already studied pricing Asian options under the square root process. Mehrdoust, Babaei and Fallah (2017) used the relation of the volatility of the square root process to that of the geometric Brownian motion to formulate a simulation algorithm of the square root process. They used this algorithm to price arithmetic Asian options. Dassios and Nagaradjasarma (2006) derived prices for Asian options using the Laplace transform. They calculated the joint moment of the square root process and its temporal integral. However, the formulas they derived are in the form of a series and are not in closed form.

In this study, we investigate the efficiency of the multilevel Monte Carlo method in pricing Asian options, where the underlying asset follows the square root process, and using the Riemann discretization scheme to approximate the involved integral. Specifically, we construct an algorithm for simulation of the square root process using its transition density and establish an upper bound of the rate of mean square convergence of the Riemann sum in approximating the temporal integral of the square root process. Lastly, we price Asian options under the square root process using the multilevel Monte Carlo method and compare its computational cost with the standard Monte Carlo method for different error bounds.

METHODS

The focus of our study is on the pricing of continuous arithmetic Asian options, which have a payoff of

$$\max\left(\frac{1}{T}\int_0^T S_u du - K, 0\right), \quad (1)$$

where S_t denotes the stock price at time t and follows the square root process, K denotes the strike price and T is the time to maturity (Boyle and Potapchik 2008). The strike price can be interpreted as the preset value of the average stock price that the buyer of the option can handle.

The multilevel Monte Carlo method is used to compute the price of the continuous arithmetic Asian options. It is fit for this scenario since a discretization scheme to compute for the integral in the payoff is also used. These methods and the distribution of the square root process are discussed in this section.

Multilevel Monte Carlo Method

The multilevel Monte Carlo is a method for discretization with the application of standard Monte Carlo. The Monte Carlo method is one of the methods used to approximate option prices. A simple summary of steps in risk-neutral options pricing using the Monte Carlo method is given in the book of Glasserman (2004), which is as follows:

1. replace the drift with the risk-free interest rate,
2. simulate paths and calculate payoff of the option derivative security on each path,
3. get the present value of discount payoffs at the risk-free rate; and
4. calculate average over paths.

The drift describes the rate of increase of the stock price and is replaced by the risk-free rate so that the options price calculated is risk-neutral.

A discretization method may be needed in step 2. If this is the case, the multilevel Monte Carlo method can be very helpful. The use of the multilevel Monte Carlo method in calculating options price can reduce the computational complexity of the simulation. This is computed by Giles (2008) and is presented in the theorem below.

Theorem. Let P denote a functional of the solution of an SDE for a given path, and let $\widehat{P}_l$ denote the corresponding approximation using a numerical discretization with time step $h_l = M^{-l}T$ where M denotes the number of Monte Carlo samples.

If there exist independent estimators $\widehat{Y}_l$ based on N_l Monte Carlo samples, and positive constants $\beta, \gamma, c_1, c_2, c_3$ and $\alpha \geq \frac{1}{2}\min(\beta, \gamma)$ such that

1. $E\left[\widehat{P}_l - P\right] \leq c_1 h_l^{\alpha}$
2. $E\left[\widehat{Y}_0\right] = E\left[\widehat{P}_0\right]$ and $E\left[\widehat{Y}_l\right] = E\left[\widehat{P}_l - \widehat{P_{l-1}}\right]$ for $l > 0$
3. $\mathrm{Var}\left[\widehat{Y}_l\right] \leq c_2 N_l^{-1} h_l^{\beta}$
4. $C_l \leq c_3 N_l h_l^{-\gamma}$, where C_l is the computational complexity of $\widehat{Y}_l$

then there exists a positive constant c_4 such that for any $\epsilon < e^{-1}$, there are values L and N_l for which the multilevel estimator, $\widehat{Y} = \sum_{l=0}^{L} \widehat{Y}_l$, has a mean-square error with bound, $E\left[\left(\widehat{Y} - E[P]\right)^2\right] < \epsilon^2$, with a computational complexity C with bound

$$C \leq \begin{cases} c_4\epsilon^{-2}, & if\ \beta > \gamma \\ c_4\epsilon^{-2}(\log(\epsilon))^2, & if\ \beta = \gamma \\ c_4\epsilon^{-2-(\gamma-\beta)/\alpha}, & if\ 0 < \beta < \gamma \end{cases}. \quad (2)$$

The theorem shows that the relationship between parameters γ and β is important to determine the computational cost of implementing the multilevel Monte Carlo method. It shows that the ideal scenario is when $\beta > \gamma$ since this gives the smallest bound. The parameter γ can be determined from the computational complexity of computing the estimators, which are approximations from using a numerical discretization. While the parameter β is related to the variance of the estimator or the mean square error of the estimator. This implies that it will be beneficial if a discretization scheme with a high order of mean square convergence will be used in approximations.

To implement the multilevel Monte Carlo method, the given algorithm based on Giles (2008) can be used**.**

1. Start with $L = 2$ and $N_l = 10^4$ for $, l = 0, \ldots, L$.
2. Estimate Var[$\hat{Y}$] using the initial set of samples.
3. Define optimal $N_l, l = 0, \ldots, L$ using $N_l = \left\lceil 2\epsilon^{-2}\sqrt{h_l^{\gamma}\mathrm{Var}[\hat{Y}_l]}\left(\sum_{l=0,}^{L}\sqrt{\mathrm{Var}[\hat{Y}_l]/h_l^{\gamma}}\right)\right\rceil$.
4. Evaluate extra samples at as needed for the new $N_l, l = 0, \ldots, L$.
5. Test for convergence using $|\hat{Y}_L| < \frac{1}{\sqrt{2}}(M^{\alpha} - 1)\epsilon$ If it is not converged, set L := L + 1 and go to step 2. Otherwise, stop.

Matlab codes found in https://people.maths.ox.ac.uk/gilesm/mlmc/#MATLAB are used and adjusted for this study. The code includes estimation of the parameters:α, β, and γ.

Reimann Discretization Scheme

Note that we consider S_t to follow the square root process. Unfortunately, $\int_0^T S_u du$ has no explicit solution. Hence, we need to integrate numerically, that is, we discretize this integral to approximate the payoff. For this study, we consider the standard integral discretization scheme which is the Riemann discretization scheme.

Let $R_T = \int_0^T S_u du$ where S_u is a square root process. Divide $[0, T]$ into N equally spaced subintervals. Let $h = T/N$ be the step size and define $t_k = kh$. Let $\widehat{R_T}$ be the Riemann Sum approximate of R_T so that

$$\widehat{R_T} = h\sum_{k=0}^{N-1} S_{t_k}. \quad (3)$$

Furthermore, we define $\widehat{R_{t_k}}$, for $k = 0, \ldots, N$ by

$$\widehat{R_{t_k}} := h\sum_{i=0}^{k-1} S_{t_i}. \quad (4)$$

Hence, we just need to be able to determine the path of the square root process.

Also, for $t \in [t_k, t_{k+1})$, we define

$$\widehat{R_t} := \widehat{R_{t_k}} + (t - t_k)S_{t_k} = \widehat{R_{t_k}} + \int_{t_k}^{t} S_{t_k} du. \quad (5)$$

Clearly, $\widehat{R_t}$ is a continuous process for $0 \leq t \leq T$.

Square Root Process

The square root process is the unique strong solution of the stochastic differential equation given by

$$dS_t = \mu S_t dt + \sigma\sqrt{S_t}dW_t, \quad (6)$$

with initial condition $S_0 = s_0 > 0$, where μ and $\sigma > 0$ are the drift and diffusion coefficient, respectively, and W_t is a Brownian motion. The drift describes the rate of increase of the process while the diffusion describes its volatility.

Equation (6) can be written as

$$S_t = S_0 + \int_0^t \mu S_u du + \int_0^t \sigma\sqrt{S_u}\, dW_u. \quad (7)$$

The square root process takes only non-negative values, that is, $S_t \geq 0$ for all $t \geq 0$. (Dassios and Nagaradjasarma 2006; Kloeden and Neuenkirch 2013)

Let $P(t, S|S_0)$ be the transition density function for the square root process, that is, the probability that at time t the value of the process is S provided that S_0 is known. The associated Kolmogorov forward equation of (6), which is used to solve for the transition density function, is given by the parabolic equation

$$\frac{\partial P}{\partial t} = \frac{\partial^2}{\partial S^2}\left(\frac{1}{2}\sigma^2 SP\right) - \frac{\partial}{\partial S}(\mu SP). \quad (8)$$

The solution to equation (8) was derived by Feller (1951), which is positivity preserving, that is, $S_t > 0$. However, S_t can be zero and it is an absorbing boundary. The solution is a norm decreasing transition density, that is, $\int_0^{\infty} P(t, S \mid S_0)\, dS < 1$ (Brecher and Lindsay 2012), and is given by

$$P(t, S \mid S_0) = \frac{2\mu}{\sigma^2(e^{\mu t} - 1)}\sqrt{\frac{e^{-\mu t}S}{S_0}}\exp\left(\frac{-2\mu(S + S_0e^{\mu t})}{\sigma^2(e^{\mu t} - 1)}\right)I_1\left(\frac{4\mu\sqrt{e^{-\mu t}SS_0}}{\sigma^2(1 - e^{-\mu t})}\right), \quad (9)$$

where $I_1(x)$ is the modified Bessel function of the first kind of order 1.

Rewriting, we obtain that

$$P(t, S \mid S_0) = \frac{4\mu}{\sigma^2(e^{\mu t} - 1)}\, p_{\chi'^2}\left(\frac{4\mu S_0}{\sigma^2(1 - e^{-\mu t})}, 4, \frac{4\mu S e^{-\mu t}}{\sigma^2(1 - e^{-\mu t})}\right), (10)$$

where $p_{\chi'^2}(x, v, \lambda)$ is the probability density function of a non-central chi-square with v degrees of freedom and non-centrality parameter λ.

Fixing t, the decumulative distribution function of S given S_0 is

$$Pr[S \geq x \mid S_0] = \int_x^{\infty} P(t, S \mid S_0)dS$$
$$= \int_x^{\infty} \frac{4\mu}{\sigma^2(e^{\mu t} - 1)}\, p_{\chi'^2}\left(\frac{4\mu S_0}{\sigma^2(1 - e^{-\mu t})}, 4, \frac{4\mu S e^{-\mu t}}{\sigma^2(1 - e^{-\mu t})}\right)dS$$
$$= \chi'^2\left(\frac{4\mu S_0}{\sigma^2(1 - e^{-\mu t})}, 2, \frac{4\mu x}{\sigma^2(e^{\mu t} - 1)}\right). \quad (11)$$

This shows that the decumulative distribution function can be written as a cumulative distribution function of the non-central chi-squared distribution. Hence, the integral of the transition density function (10) over the permissible values of S can be written as

$$\int_0^{\infty} P(t, S \mid S_0)dS = \int_0^{\infty} \frac{4\mu}{\sigma^2(e^{\mu t} - 1)} p_{\chi'^2}\left(\frac{4\mu S_0}{\sigma^2(1 - e^{-\mu t})}, 4, \frac{4\mu S e^{-\mu t}}{\sigma^2(1 - e^{-\mu t})}\right)dS$$
$$= \chi^2\left(\frac{4\mu S_0}{\sigma^2(1 - e^{-\mu t})}, 2\right)$$

$$= 1 - \exp\left(\frac{-2\mu S_0}{\sigma^2(1-e^{-\mu t})}\right). \quad (12)$$

The transition density is defective since this solution is norm decreasing, that is, $\int_0^\infty P(t,S \mid S_0)dS < 1$. The quantity $\exp\left(\frac{-2\mu S_0}{\sigma^2(1-e^{-\mu t})}\right)$ represents the probability that the process is absorbed at $S_t = 0$. This can be interpreted as the probability of default or bankruptcy. The full transition density should account this absorption. Hence,

$$P_{full}(t,S \mid S_0) = \begin{cases} \exp\left(\frac{-2\mu S_0}{\sigma^2(1-e^{-\mu t})}\right), & \text{if } S_t = 0 \\ \frac{4\mu}{\sigma^2(e^{\mu t}-1)} p_{\chi'^2}\left(\frac{4\mu S_0}{\sigma^2(1-e^{-\mu t})}, 4, \frac{4\mu S e^{-\mu t}}{\sigma^2(1-e^{-\mu t})}\right), & \text{otherwise.} \end{cases} \quad (13)$$

RESULTS AND DISCUSSION

This section comprises the results of our study which aims to price Asian options under the square root process using the multilevel Monte Carlo method with the use of the Riemann discretization scheme. An algorithm to simulate the values of the square root process needed to implement the Riemann discretization scheme is created. The order of the mean square convergence of the Riemann scheme is examined to have an idea of the efficiency of the use of the multilevel Monte Carlo. Finally, we price Asian options using different sets of parameters and analyze the computational complexity of using the multilevel Monte Carlo method.

Simulation of the Square Root Process

Recall that a path of the square root process is needed for the Riemann discretization scheme as seen in equation (3). However, the square root process has no closed form solution. Hence, we simulate the path of the square root process using its transition density function.

The full transition density of the process S_t and its decumulative distribution function, given by $\chi'^2\left(\frac{4\mu S_0}{\sigma^2(1-e^{-\mu t})}, 2, \frac{4\mu S_t}{\sigma^2(e^{\mu t}-1)}\right)$ for $S_t > 0$, can be used to have a simulation of the path of S_t. This can be done by using the inverse transform method of sampling but instead of using the cumulative function, we used the decumulative function and utilized the properties of the chi-square random variable. To do this, fixing t, we draw y from a uniform distribution on $(0,1)$. y represents $P[S \geq S_t \mid S_0]$. However, we will not directly use y to sample S_t, instead, we will use it to determine if S_t will be trapped at 0, that is, $S_u = 0$ for $u \geq t$. Denote

$$y_{max} := \int_0^\infty P(t,S \mid S_0)dS = 1 - \exp\left(\frac{-2\mu S_0}{\sigma^2(1-e^{-\mu t})}\right). \quad (14)$$

Hence if $y > y_{max}$, this would imply that S_t will be trapped at 0. On the other hand, if $y \leq y_{max}$, $S_t > 0$ and we can use (11) to sample S_t.

If $A \sim \chi_2'^2(\lambda)$ then $A = B + C$, where B follows the non-central chi-squared distribution with 2 degrees of freedom and C follows a chi-squared distribution with 1 degree of freedom, as mentioned in (Johnson, Kotz, and Balakrishnan 1995). This relationship implies that to generate A, it suffices to generate C and an independent standard normal random variable Z and set $A = (Z+D)^2 + C$, where $D^2 = \lambda$. Note that $C \sim \chi_1^2$. Hence, $C = X^2$, where $X \sim N(0,1)$, that is, X is a standard normal random variable. Thus, we have

$$A = (Z+D)^2 + X^2, \quad (15)$$

where $X \sim N(0,1)$, $Z \sim N(0,1)$ and $D^2 = \lambda$.

Note that from (11), we want to sample the parameter λ of a non-central chi-squared distribution with 2 degrees of freedom. Thus, the sampling algorithm for S_t can be summarized as follows:

1. Generate y from a uniform distribution on $(0,1)$.
2. If $y > y_{max} = 1 - \exp\left(\frac{-2\mu S_0}{\sigma^2(1-e^{-\mu t})}\right)$, set $S_t = 0$ and stop. Otherwise, go to 3.
3. Generate x from $X \sim N(0,1)$ and z from $Z \sim N(0,1)$.
4. If $\left(\frac{4\mu S_0}{\sigma^2(1-e^{-\mu t})} - x^2\right) \geq 0$, set $d = \left(\frac{4\mu S_0}{\sigma^2(1-e^{-\mu t})} - x^2\right)^{1/2} - z$. Otherwise, go back to step 3.
5. Set $S_t = d^2\left(\frac{\sigma^2(e^{\mu t}-1)}{4\mu}\right)$ and stop.

Fixing t, the expectation and variance of S_t can also be calculated.

The expectation of S_t, denoted by μ_S, is given by

$$\mu_S = E[S_t|S_0] = \int_0^\infty S\, P_{full}(t,S \mid S_0)dS = \int_0^\infty S\, P(t,S \mid S_0)dS. \quad (16)$$

Observe that

$$\begin{aligned} &SP(t,S \mid S_0) \\ &= S\frac{4\mu}{\sigma^2(e^{\mu t}-1)} p_{\chi'^2}\left(\frac{4\mu S_0}{\sigma^2(1-e^{-\mu t})}, 4, \frac{4\mu S e^{-\mu t}}{\sigma^2(1-e^{-\mu t})}\right) \\ &= S_0 e^{\mu t}\frac{4\mu}{\sigma^2(e^{\mu t}-1)} p_{\chi'^2}\left(\frac{4\mu S e^{-\mu t}}{\sigma^2(1-e^{-\mu t})}, 4, \frac{4\mu S_0}{\sigma^2(1-e^{-\mu t})}\right). \quad (17) \end{aligned}$$

Hence,

$$\mu_S = S_0 e^{\mu t}. \quad (18)$$

We validate the simulation algorithm by comparing the analytic mean from the simulated mean using 100,000 simulation runs as shown in Table 1.

Mean Square Convergence of the Riemann Discretization Scheme

We first observe the second moment of the square root process. Using the mean of a non-central chi-squared distributed random variable and (10), and for known S_t and $u \geq t$, we have

$$\begin{aligned} E[S_u^2 \mid S_t] &= \int_0^\infty S^2 P_{full}(u,S \mid S_t)dS \\ &= \int_0^\infty S^2 P(u,S \mid S_t)dS \\ &= S_t e^{\mu u}\frac{\sigma^2(1-e^{-\mu u})}{4\mu e^{-\mu u}}\left(4 + \frac{4\mu S_t}{\sigma^2(1-e^{-\mu u})}\right) \\ &= S_t e^{2\mu u}\sigma^2(1-e^{-\mu u}) + S_t^2 e^{2\mu u}. \quad (19) \end{aligned}$$

Proposition 1. For a square root process, with S_t known and $u - t < \frac{1}{2\mu}$, where μ is the drift coefficient of the process, there exists a constant $C > 0$ such that

$$E[S_u^2 - S_t^2] < C2\mu(u-t). \quad (20)$$

Proof is presented in Appendix A.

Table 1: Comparison of the analytic mean to the simulated mean with $S_0 = 100$

t	μ	σ	E[S_t]	Simulated Mean	Relative Difference
1	0.05	0.01	105.1271	105.1269	1.99413×10^{-6}
1	0.05	0.05	105.1271	105.1277	5.6157×10^{-6}
1	0.05	0.5	105.1271	105.1201	6.66777×10^{-5}
1	0.1	0.05	110.5171	110.5174	2.78864×10^{-6}
1	0.05	0.4	105.1271	105.1282	1.03718×10^{-5}
1	0.1	0.5	110.5171	110.5098	6.5979×10^{-5}
2	0.05	0.5	110.5171	110.5392	0.0002
2	0.1	0.05	122.1408	122.1405	1.83546×10^{-6}
2	0.05	0.4	110.5171	110.5062	9.85532×10^{-5}
3	0.05	0.01	116.1834	116.1836	1.5025×10^{-6}
3	0.1	0.05	134.9859	134.9841	1.31922×10^{-5}
3	0.1	0.5	134.9859	135.0176	0.000235
4	0.05	0.01	122.1403	122.1399	3.07692×10^{-6}
4	0.05	0.05	122.1403	122.1378	2.02703×10^{-5}
4	0.1	0.5	149.1825	149.1601	0.000145
5	0.05	0.05	128.4025	128.4021	3.43972×10^{-6}
5	0.1	0.05	164.8721	164.8767	2.77362×10^{-5}
5	0.05	0.4	128.4025	128.4145	9.31316×10^{-5}

Table 2: Asian options price comparison from the paper of F. Mehrdoust, et al.

S_0	K	T	μ	σ	Price (Merdoust, et al)	Price (MLMC)	Relative Difference
100	95	1	0.09	0.05	8.788	8.809	0.0024
100	100	1	0.09	0.05	4.322	4.239	0.0192

This will be useful in examining the mean square convergence of the Riemann sum where the integrand is the square root process.

Proposition 2. Let $h < \frac{1}{2\mu}$. There exists a constant $\kappa > 0$ independent of h such that

$$E\left(\sup_{t\in[0,T]} \left|\widehat{R_t} - R_t\right|^2\right) \leq \kappa h^2. \quad (21)$$

Proof is presented in Appendix B.

Thus, this shows that the mean square convergence of the Riemann scheme as the discretization scheme for the temporal integral of the square root process is at least of order 1. Hence, β, an important parameter in equation (2), should be at least 2.

Multilevel Monte Carlo Simulation

We now implement the multilevel Monte Carlo method. Note that the Riemann discretization scheme, with the use of the algorithm for simulating the path of the square root process, is used to simulate the payoff needed in the implementation of the multilevel Monte Carlo method.

Since the number of function evaluation needed to implement the Riemann scheme is just proportional to the number of time steps, γ should be approximately close to 1. From the previous subsection, β should also be at least 2. These give us a reason to use the multilevel Monte Carlo method so that the computational cost is less.

Comparison of the computational costs of the multilevel Monte Carlo method and the standard Monte Carlo method for a set of parameters is shown in Figure 1. As the error bound, ϵ, decreases, computational advantage of multilevel Monte Carlo method over standard Monte Carlo increases drastically. This can be attributed to the high value of β which should lead to more computational savings as written in the complexity theorem.

The price of Asian options obtained from the simulations, with $\epsilon = 0.0001$, are also compared to results from the study of Mehrdoust et al (2017), as shown in Table 2, and from the study

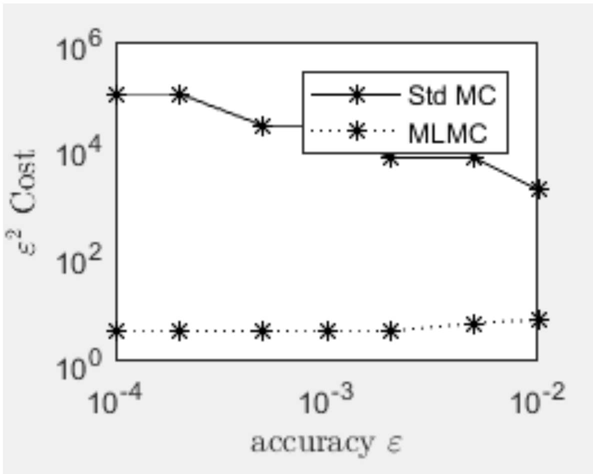

Figure 1: Computational costs given $S_0 = 100$, $K = 95$, $T = 1$, $\mu = 0.09$ and $\sigma = 0.05$

of Dassios and Nagaradjasarma (2006), as shown in Table 3. Even though the method we used were different from the methods used by these authors, the computed prices we obtained are close to the computed prices from their work, considering the presented parameter values.

From the simulation results, we can say that using the multilevel Monte Carlo method, with Riemann scheme and the simulation algorithm for the square root process, to price Asian options under the square root process is efficient as compared to the standard Monte Carlo method and gives reliable results.

CONCLUSION

This study gives an alternative way of pricing Asian options whose underlying asset follows the square root process. We were able to use an efficient simulation algorithm based on the transition density to determine the values of the square root process at a desired specified time. More importantly, this study computed prices of Asian options under the square root process using the multilevel Monte Carlo method and it has shown that the use of this method in this context is effective in terms of reducing computational cost. We have shown that the mean square convergence of the Riemann scheme as the discretization scheme for the temporal integral is at least of order 1, which has been an important factor in the reduction of computational cost.

This study only focused on the use of multilevel Monte Carlo in pricing the discussed Asian option. It is recommended to determine the advantage of using the multilevel Monte Carlo method in this context as compared to analytic methods and other simulation methods.

Only the Riemann scheme was used and analyzed in this study since it is the standard numerical method in discretizing integrals. Other integral discretization schemes with higher order of accuracy can also be used. It is expected that these will improve the method discussed here. Hence, their convergences and effects in the reduction of computational cost are interesting extensions of this work. The study on the weak convergences can also be conducted although this is not useful when analyzing the computational cost of the multilevel Monte Carlo method.

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CONFLICT OF INTEREST

We report no conflict of interest.

CONTRIBUTION OF INDIVIDUAL AUTHORS

All authors contributed to conceptualizing the study. Kabiri did the derivations and simulations and prepared the manuscript. David reviewed and gave comments to improve the study.

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Appendix A: Proof of Proposition 1

$$E[S_u^2 - S_t^2] = E[S_u^2 \mid S_t] - S_t^2$$

$$= S_t e^{2\mu(u-t)} \sigma^2 \left(1 - e^{-\mu(u-t)}\right) + S_t^2 e^{2\mu(u-t)}$$

$$- S_t^2 \; using \; (14)$$

$$= e^{2\mu(u-t)} (S_t \sigma^2 + S_t^2) - e^{\mu(u-t)} S_t \sigma^2 - S_t^2$$

$$= \sum_{i=0}^{\infty} \frac{\left(2\mu(u-t)\right)^i}{i!} (S_t \sigma^2 + S_t^2)$$

$$- \sum_{i=0}^{\infty} \frac{\left(\mu(u-t)\right)^i}{i!} S_t \sigma^2 - S_t^2$$

$$= \sum_{i=1}^{\infty} \left[\frac{\left(2\mu(u-t)\right)^i}{i!} (S_t \sigma^2 + S_t^2) \right.$$

$$\left. - \frac{\left(\mu(u-t)\right)^i}{i!} S_t \sigma^2 \right]$$

$$< \sum_{i=1}^{\infty} \left[\frac{\left(2\mu(u-t)\right)^i}{i!} (S_t \sigma^2 + S_t^2) \right]$$

Since $u - t < \frac{1}{2\mu}$, then $2\mu(u - t) < 1$. Hence, there exists a constant C such that

$$E[S_u^2 - S_t^2] < C 2\mu(u - t),$$

as desired.

Appendix B: Proof of Proposition 2

Let $\epsilon_t = R_t - \widehat{R_t}$, $t \in [0,T]$. Note that if $t \in [0,T]$ then $t \in [t_k, t_{k+1}]$ where k can take the values $0,1,2, \dots N-1$.

Hence,

$$\epsilon_t = \left(R_{t_k} + \int_{t_k}^{t} S_u du\right) - \left(\widehat{R_{t_k}} + \int_{t_k}^{t} S_{t_k} du\right)$$

$$= \epsilon_{t_k} + \int_{t_k}^{t} (S_u - S_{t_k}) du$$

$$= \epsilon_{t_k} + \int_{t_k}^{t} \left(\int_{t_k}^{u} \mu S_v dv + \int_{t_k}^{u} \sigma S_v^{1/2} dW_v\right) du$$

$$= \epsilon_{t_k} + \int_{t_k}^{t}\int_{t_k}^{u} \mu S_v dv\, du + \int_{t_k}^{t}\int_{t_k}^{u} \sigma S_v^{1/2} dW_v du\,.$$

$$= \epsilon_{t_k} + \int_{t_k}^{t}\int_{v}^{t} \mu S_v du\, dv + \int_{t_k}^{t}\int_{v}^{t} \sigma S_v^{1/2} du\, dW_v$$

$$= \epsilon_{t_k} + \int_{t_k}^{t} (t-v)\mu S_v dv + \int_{t_k}^{t} (t-v)\sigma S_v^{1/2} dW_v.$$

Applying Ito's formula to ϵ_t^2, we obtain

$$\epsilon_t^2 = \epsilon_{t_k}^2 + \int_{t_k}^{t} 2\,\epsilon_v (t-v)\mu S_v dv + \int_{t_k}^{t} 2\,\epsilon_v (t-v)\sigma S_v^{1/2} dW_v + \int_{t_k}^{t} (t-v)^2\sigma^2 S_v dv.$$

Computing the expectation and noting that the expectation of an Ito integral is zero, we have

$$E[\epsilon_t^2] = E[\epsilon_{t_k}^2] + E\left[\int_{t_k}^{t} 2\,\epsilon_v (t-v)\mu S_v dv\right] + E\left[\int_{t_k}^{t} (t-v)^2\sigma^2 S_v dv\right]$$

$$\leq E[\epsilon_{t_k}^2] + E\left[\int_{t_k}^{t} 2 \mid \epsilon_v \mid (t-v)\mu S_v dv\right] + E\left[\int_{t_k}^{t} (t-v)^2\sigma^2 S_v dv\right].$$

Note that $\mid \epsilon_v \mid 2(t-v)S_v \leq \frac{|\epsilon_v|^2}{2} + \frac{4(t-v)^2 S_v^2}{2}$ by Young's inequality. Thus,

$$E[\epsilon_t^2] \leq E[\epsilon_{t_k}^2] + E\left[\int_{t_k}^{t} \mu \frac{|\epsilon_v|^2}{2} dv\right] + E\left[\int_{t_k}^{t} \mu 2(t-v)^2 S_v^2 dv\right] + E\left[\int_{t_k}^{t} (t-v)^2\sigma^2 S_v dv\right]$$

$$= E[\epsilon_{t_k}^2] + \frac{\mu}{2}E\left[\int_{t_k}^{t} |\epsilon_v|^2 dv\right] + E\left[\int_{t_k}^{t} 2\mu (t-v)^2 S_v^2 dv\right] + E\left[\int_{t_k}^{t} (t-v)^2\sigma^2 S_v dv\right]$$

$$= E[\epsilon_{t_k}^2] + \frac{\mu}{2}E\left[\int_{t_k}^{t} |\epsilon_v|^2 dv\right] + E\left[\int_{t_k}^{t} (t-v)^2 (2\mu S_v^2 + \sigma^2 S_v) dv\right]$$

$$\leq E[\epsilon_{t_k}^2] + \frac{\mu}{2}E\left[\int_{t_k}^{t} |\epsilon_v|^2 dv\right] + E\left[\int_{t_k}^{t} (t-t_k)^2 (2\mu S_v^2 + \sigma^2 S_v) dv\right]$$

$$= E[\epsilon_{t_k}^2] + \frac{\mu}{2}E\left[\int_{t_k}^{t} |\epsilon_v|^2 dv\right] + (t-t_k)^2 E\left[\int_{t_k}^{t} (2\mu S_v^2 + \sigma^2 S_v) dv\right].$$

Applying the Ito's formula to S_v^2, we obtain

$$S_t^2 = S_{t_k}^2 + \int_{t_k}^{t} 2\,\mu S_v^2 dv + \int_{t_k}^{t} 2\,\sigma S_v \sqrt{S_v} dW_v + \int_{t_k}^{t} \sigma^2 S_v dv$$

$$= S_{t_k}^2 + \int_{t_k}^{t} 2\,\sigma S_v \sqrt{S_v} dW_v + \int_{t_k}^{t} (2\mu S_v^2 + \sigma^2 S_v) dv.$$

Then,

$$\int_{t_k}^{t} (2\mu S_v^2 + \sigma^2 S_v) dv = S_t^2 - S_{t_k}^2 - \int_{t_k}^{t} 2\,\sigma S_v \sqrt{S_v} dW_v.$$

Thus,

$$E\left[\int_{t_k}^{t} (2\mu S_v^2 + \sigma^2 S_v) dv\right] = E\left[S_t^2 - S_{t_k}^2\right].$$

Hence, by the previous proposition,

$$E[\epsilon_t^2] \leq E[\epsilon_{t_k}^2] + \frac{\mu}{2}E\left[\int_{t_k}^{t} |\epsilon_v|^2 dv\right] + (t-t_k)^2 E\left[S_t^2 - S_{t_k}^2\right]$$

$$< E[\epsilon_{t_k}^2] + \frac{\mu}{2}E\left[\int_{t_k}^{t} |\epsilon_v|^2 dv\right] + (t-t_k)^2 C 2\mu (t-t_k)$$

$$\leq E[\epsilon_{t_k}^2] + \frac{\mu}{2}E\left[\int_{t_k}^{t} |\epsilon_v|^2 dv\right] + 2C\mu h^3$$

$$= E[\epsilon_{t_k}^2] + \frac{\mu}{2}E\left[\int_{t_k}^{t} |\epsilon_v|^2 dv\right] + 2C\mu h^3.$$

Applying Gronwall's inequality, we obtain

$$E[\epsilon_t^2] < \left(E[\epsilon_{t_k}^2] + 2C\mu h^3\right) e^{\frac{\mu}{2}\int_{t_k}^{t} dv}$$

$$= \left(E[\epsilon_{t_k}^2] + 2C\mu h^3\right) e^{\frac{\mu}{2}(t-t_k)}$$

$$\leq \left(E[\epsilon_{t_k}^2] + 2C\mu h^3\right) e^{\frac{\mu}{2}h}.$$

If $t = t_{k+1}$, then $E[\epsilon_{t_{k+1}}^2] \leq \left(E[\epsilon_{t_k}^2] + 2C\mu h^3\right) e^{\frac{\mu}{2}h}$.

Since $E[\epsilon_{t_0}^2] = 0$ and $x_{n+1} \leq a x_n + b$ implies that $x_n \leq a^n x_0 + n e^{n(a-1)} b$ for $a \geq 1$ and $b > 0$,

$$E[\epsilon_{t_k}^2] < k e^{k\left(e^{\frac{\mu}{2}h} - 1\right)} 2C\mu h^3 e^{\frac{\mu}{2}h}.$$

Then,

$$E[\epsilon_t^2] < \left(k e^{k\left(e^{\frac{\mu}{2}h} - 1\right)} 2C\mu h^3 e^{\frac{\mu}{2}h} + 2C\mu h^3\right) e^{\frac{\mu}{2}h}.$$

Using Doob's L^2-inequality,

$$E\left[\sup_{t\in[0,T]} |\epsilon_t|^2\right] \leq 4 \sup_{t\in[0,T]} E[\epsilon_t^2]$$

$$< 4\left(N e^{N\left(e^{\frac{\mu}{2}h} - 1\right)} 2C\mu h^3 e^{\frac{\mu}{2}h} + 2C\mu h^3\right) e^{\frac{\mu}{2}h}.$$

Since $h < \frac{1}{2\mu}$, then $2\,\mu h < 1$ which implies that $\frac{\mu}{2}h < \frac{1}{4}$. Hence, we may use the fact that $e^{\frac{\mu}{2}h} = 1 + O\left(\frac{\mu}{2}h\right)$. Thus, we obtain

$$E\left[\sup_{t\in[0,T]} |\epsilon_t|^2\right] < 4\left[Ne^{N\left(O\left(\frac{\mu}{2}h\right)\right)} 2C\mu h^3 \left(1 + O\left(\frac{\mu}{2}h\right)\right) + 2C\mu h^3\right]\left(1 + O\left(\frac{\mu}{2}h\right)\right).$$

Noting that $h = \frac{T}{N}$, there exists a constant κ independent of h, such that,

$$E\left[\sup_{t\in[0,T]} |\epsilon_t|^2\right] \leq \kappa h^2.$$

Symbol/Function	Meaning
S_t	- the square root process which denotes the stock price at time t
K	- strike price of the option
T	- time to maturity of the option
h	- time step used in numerical methods
l	- level in the multilevel Monte Carlo method
N_l	- number of Monte Carlo samples at level l
β	- parameter related to the order of mean square convergence or variance of the estimator per level
γ	- parameter related to the rate of the computational complexity of the estimator per level
$\widehat{R_t}$	- Riemann sum approximate over the interval from 0 to t
μ	- drift coefficient of the square root process
σ	- diffusion coefficient of the square root process
W_t	- Brownian motion
ϵ	- error bound
max(x)	- gives the maximum element of the array x
$E[x]$	- gives the expected value of x
$\mathrm{Var}[x]$	- gives the variance of x
$Pr[x\|y]$	- conditional probability (gives the probability of x given y)
$N(0,1)$	- normal distribution with mean zero and variance of one
$\chi_v'^2(\lambda)$	- non-central chi-squared distribution with v degrees of freedom and non-centrality parameter λ
χ_v^2	- chi-squared distribution with v degrees of freedom
$\exp(x)$	- equal to e^x
$\sup X$	- gives the smallest upper bound of the set X