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Solution to the Three-Player Gambler's Ruin with Variable Bet Size Using Recursions Based on Multigraphs

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ABSTRACT

This paper studies the three-player gambler's ruin with variable bet size, that is, more than one chip may be transferred from one player to another. Weighted directed multigraphs were constructed to model the transitions between chip states. Linear systems are constructed based on the connections between nodes in these graphs. Solutions for the placing probabilities of each player are obtained from these linear systems. Expected time until ruin is solved by modeling the game as a Markov process. A numerical algorithm is developed to solve the No Limit three-player gambler's ruin problem for any positive integer chip total.

INTRODUCTION

In the two-player gambler's ruin problem, two gamblers with initial wealths A and B play an even-money game against each other, each time betting a fixed amount. From the first gambler's perspective, this scenario is equivalent to the classic gambler's ruin problem with target wealth $S = A+B$. Their probability of winning is $\frac{A}{A+B}$, which was proved using discrete-time (random walk) methods or continuous time (Brownian motion) methods [9]. The expected game duration is $T = AB$ [8], which was obtained using difference equations and recursions.

The three-player gambler's ruin problem, first studied by Bachelier in 1912 [2], has several variations based on the players involved in each betting round, and how winners and losers are selected in these rounds. One form of the three-player game is called the three-tower problem. In each round, a game is played with one winner and one loser. Suppose that each game involves exactly two players, each with an equal probability of winning the game. The players are paired randomly with equal probabilities, so that each pair has a probability 1/3 of being selected. If we fix the bet size to 1 unit and the initial wealths are natural numbers, we get the three-tower game, which we can also refer to as the gambler's ruin model with unit bets.

If the bet size may vary depending on the wealths of the players, the scenario is defined as the No Limit variation, or "occasionally all in" as described in a paper by Diaconis and Ethier [6]. The expected game duration for the three-tower problem has been solved and is given by $T = \frac{3ABC}{A+B+C}$ [2, 7, 13]. Stirzaker introduced solutions using martingales in this area of research [13].

In the past years, several solutions have been used in modeling the three-tower problem. In these models, the probability of a player's success is easily solved by recursion and is given by the proportion of a player's wealth to the total wealth of all players involved in the game. Bruss et al. (2002) used martingales in

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giving an asymptotic solution to the three-tower problem [3]. David (2015) used weighted directed multigraphs and recursions to model and solve the three-tower problem [4]. For a fixed wealth S, unique states were generated and transitions between these states were represented as edges in a multigraph. A linear system was constructed based on recurrence equations, which was then solved to obtain the ruin probabilities of each player. After ruin probabilities are solved, placing probabilities of each player can be computed easily. Diaconis et al. (2020) studied gambler's ruin estimates on finite inner uniform domains, where they studied the three-player problem as one example [5]. They related these estimates to properties of the underlying Markov chain and its Doob transform. Diaconis and Ethier (2020) discussed six different methods of approximating gambler's ruin probabilities [6]. They mentioned that while exact computations using Markov chain methods are feasible for wealth totals which are not too large, these methods seem difficult for values of practical interest. Their preferred method is linear interpolation yielding numerical results accurate to about four to five decimal places.

The main purpose of this paper is to solve the placing probabilities of each player for the three-player gambler's ruin problem with varying bet amount. Specifically, the paper aims to accomplish the following. First, we will construct a weighted directed multigraph for the No Limit three-player gambler's ruin with prescribed total wealth S. From this, we will construct an appropriate linear system that represents the multigraph for the No Limit three-player gambler's ruin with the prescribed total wealth S. We will then develop a numerical algorithm for solving placing probabilities for the No Limit three-player gambler's ruin with the prescribed total wealth S. Lastly, we will present a method for calculating the expected time until ruin for the No Limit three-player gambler's ruin with the prescribed total wealth S using the previously described numerical algorithm.

THEORETICAL FRAMEWORK

Weighted Directed Multigraphs

We use the following definitions to define a weighted directed multigraph.

1. A **graph** $G(V,E)$ consists of two types of elements, namely **vertices** and **edges**. Every edge has two endpoints in the set of vertices. V is the set of vertices while E is the set of edges.
2. A **weighted graph** is a graph having a weight, or number, associated with each edge.
3. A **directed graph** is a graph where all edges are directed from one vertex to another.
4. A **multigraph** is a graph which may have multiple edges connecting the same pair of vertices.
5. A **weighted directed multigraph** is a graph satisfying items (1) to (4).

Weighted directed multigraphs will be used to model the transitions between states given the players' initial wealths. Variables for the players' placing probabilities will be assigned to each unique state. The system of equations relating these variables will then be solved for these probabilities.

Absorbing Markov Chains

We define an absorbing Markov chain using the following definitions [10].

1. Given a set of states $S = \{s_1, s_2, \ldots, s_n\}$, a **Markov chain** is a process satisfying the following conditions:
   (a) The transition probability p_{ij} of moving from an initial state s_i to any next state s_j depends only on the current state s_i and the next state s_j.
   (b) The sum of all transition probabilities from any state s_i is equal to 1, i.e.,
   $$\sum_{j=1}^{n} p_{ij} = 1 \quad (1)$$
   for all $i \in \{1,2,\ldots,n\}$.

2. A state s_i of a Markov chain is called an **absorbing state** if it is impossible to leave it, i.e., $p_{ii} = 1$.

3. An **absorbing Markov chain** is a Markov chain satisfying the following conditions:
   (a) It has at least one absorbing state.
   (b) It is possible to go from any state to at least one absorbing state in a finite number of steps.

4. A state s_i of an absorbing Markov chain is called a **transient state** if it is not absorbing.

5. Absorbing Markov chains can be represented using matrices whose sizes depend on the number of transient and absorbing states. Suppose there are t transient states and r absorbing states for an absorbing Markov chain. The transition matrix has the following **canonical form** [10]:
   $$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{Q} & \boldsymbol{R} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \quad (2)$$
   where $\mathbf{Q}$ is a $t \times t$ matrix, $\mathbf{R}$ is a nonzero $t \times r$ matrix, $\mathbf{0}$ is an $r \times t$ zero matrix, and $\mathbf{I}$ is an $r \times r$ identity matrix. The first t states are the transient states while the last r states are the absorbing states.

6. For an absorbing Markov chain $\mathbf{P}$, the matrix $\mathbf{N} = (\mathbf{I} - \mathbf{Q})^{-1}$ is called the **fundamental matrix** for $\mathbf{P}$ [10]. The entry n_{ij} of $\mathbf{N}$ gives the expected number of times that the process is in the transient state s_j if it started in the transient state s_i.

The No Limit three-player gambler's ruin will be modeled using an absorbing Markov chain in order to calculate the expected time until ruin, as shown in Section 2.

MATERIALS AND METHODS

This paper aims to calculate placing probabilities for the three players given their initial wealths. By a theorem by Bachelier [2], first place probabilities are the proportions of the players' initial wealths to the total wealth of all players, which our results will show that this is indeed the case. What remains is to calculate the probabilities of placing second and third for each player. These calculations are done using a numerical algorithm (executed in Matlab) described in this section. We first present the following.

Definition 1 *A chip state is an ordered triple (x,y,z), where $x,y,z \in \mathbb{N} \cup \{0\}$ and $x \geq y \geq z$. The coordinates of a chip state represent the wealths of each player at a given time.*

Definition 2 *A chip position is a coordinate of a chip state (x,y,z). Each position in a chip state has corresponding probabilities of finishing first, second or third among the three players.*

Definition 3 *A terminal state is a chip state where at least one chip position is zero. Terminal states have known placing probabilities for each player.*

Definition 4 *A nonterminal state is a chip state with three positive chip positions.*

Definition 5 *In a betting round involving players with wealths x and y, the permissible bet sizes are elements of the set* $\{1,2,...,\min\{x,y\}\}$*, or natural numbers less than or equal to the number of chips in the smaller stack involved in the betting round.*

Two players are selected randomly using a uniform distribution, and these two players will face each other in an even-money betting round. Without loss of generality, let the two players be Players 1 and 2 having wealths x and y, respectively. The bet size n is selected randomly using a uniform distribution from the set of permissible bet sizes $\{1,2,...,\min\{x,y\}\}$. The winner of the round is selected randomly, and that player adds to his stack n chips taken from the other player's initial stack. For example, if Player 1 wins over Player 2, their new chip stacks will be $x + n$ and $y - n$, respectively. Upon reaching a terminal state, a chip state of the form $(a,b,0)$, second place probabilities can be solved trivially. Upon reaching nonterminal states, the process is repeated.

We use the following definition for the mapping for the No Limit three-player gambler's ruin.

Definition 6 *Suppose* (x,y,z) *is a nonterminal state. The possible states after one round of betting are shown by the following map:*

$$(x,y,z) \to \begin{cases} (x+n, y-n, z), & \text{where } n \in \{1,2,\dots,\min\{x,y\}\} \\ (x-n, y+n, z), & \textit{where } n \in \{1,2,\dots,\min\{x,y\}\} \\ (x-n, y, z+n), & \textit{where } n \in \{1,2,\dots,\min\{x,z\}\} \\ (x+n, y, z-n), & \textit{where } n \in \{1,2,\dots,\min\{x,z\}\} \\ (x, y-n, z+n), & \textit{where } n \in \{1,2,\dots,\min\{y,z\}\} \\ (x, y+n, z-n), & \textit{where } n \in \{1,2,\dots,\min\{y,z\}\} \end{cases}$$

with each edge having weight $\frac{1}{6m}$*, where m is the maximum permissible bet size between the players participating in the round.*

Construction of Multigraph

Let S be the total wealth of the three players. We construct the weighted directed multigraph using the following algorithm:

1. Unique states (x,y,z) are generated such that $S = x + y + z$ and $x \geq y \geq z$, that is, the total wealth of the three players is S and their wealths are arranged in decreasing order. The generated unique states will serve as the nodes in the graph. For easy visualization, states will be arranged in columns according to their third coordinates.

2. Terminal states (which are those with a zero position) are put on the leftmost column in the graph, with the states arranged in decreasing order of their first coordinates.

3. States with the last coordinate equal to 1 are then put on the next column in the graph, with the states also arranged in decreasing order of their first coordinates. States in different columns are put in the same row if they have the same first coordinates.

4. The process of constructing rows and columns of nodes is repeated until all states are exhausted.

5. Given a nonterminal state (x,y,z), if it is possible for the wealths to become (a,b,c) after one round of betting, an edge directed from (x,y,z) to (a,b,c) is constructed, with weight equal to one-sixth multiplied to the reciprocal of the maximum bet size allowed between the two players involved in the round. The total weight of all outward edges from a nonterminal state must be 1.

6. Loops, which are edges whose initial and final nodes are the same, may be constructed if a state goes to itself (up to permutation of coordinates) after one round. Multiple edges between a pair of nodes may also be constructed if there are multiple possible ways of doing so.

Construction of Linear System

We construct the linear system using the following algorithm: [12]

1. *Variable assignment*
   A variable will be associated to each of the unique chip positions from all terminal and nonterminal states in the order they are generated.

2. *Construction of transition matrix* **Q**
   An $n \times n$ matrix **Q** is constructed, where n is the number of unique chip positions from all nonterminal states. Matrix **Q** is the matrix representing the transitions between nonterminal chip positions.

3. *Construction of transition matrix* **R**
   An $n \times m$ matrix **R** is also constructed, where m is the number of unique chip positions from all terminal states. This matrix represents the transitions from nonterminal to terminal chip positions.

4. *Computation of entries of matrices* **Q** *and* **R**
   For each of the n variables corresponding to nonterminal positions, we determine where the corresponding chip positions are being moved.

5. *Set up of linear system*
   We set up and solve the linear system

   $$(\mathbf{I} - \mathbf{Q})\mathbf{B} = \mathbf{R} \tag{4}$$

   where **B** is an $n \times m$ matrix containing the probabilities for the n unique nonterminal chip positions of ending up in the m unique terminal chip positions.

6. *Computation of placing probabilities*
   After solving for matrix **B**, we construct an $m \times 3$ matrix **W** containing the placing probabilities for the corresponding terminal positions in the three-player game. Lastly, we find **BW**, which contains the final placing probabilities.

Given total wealth S, it can be shown using integer partitions [1] and recurrence relations that $n = \left\lfloor \frac{(S-1)^2}{4} \right\rfloor$ and $m = S - 1 + \left\lfloor \frac{S}{2} \right\rfloor$. The derivation of these results is not provided in this paper.

Calculation of Expected Time until Ruin

In the three-player gambler's ruin, time until ruin is the number of games played by the players until one of them reaches zero wealth. In calculating the expected time until ruin, we use absorbing Markov chains. Recall that an absorbing Markov chain **P** has the following canonical form:

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{Q} & \boldsymbol{R} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \tag{5}$$

where **Q** represents the transitions between transient states, **R** represents the transitions from transient states to absorbing states, and **I** represents the transitions between absorbing states. In the three-player gambler's ruin, the transient states are the nonterminal states, while the absorbing states are the terminal states, where only two players have nonzero wealths. Hence, for the canonical form of the matrix, we use matrices **Q** and **R** as constructed in Section 2.2, while **I** is an m-by-m identity matrix. In modeling this problem as an absorbing Markov chain, the chain is absorbed when one of the three players is ruined. Hence, the expected time until ruin is equal to the absorption time of the chain. We use the fundamental matrix **N** of the absorbing Markov chain **P**, with the existence of the former guaranteed by the following proposition [10].

Proposition 1 *Let* **P** *be an absorbing Markov chain, and* **Q** *the submatrix of* **P** *representing the transitions between transient states. Then* $\mathbf{N} = (\mathbf{I} - \mathbf{Q})^{-1}$ *exists.*

Proposition 1 also guarantees the existence of solutions for our linear system for ruin probabilities. We now use the following theorem in finding the expected time until ruin [10].

Theorem 2 *Given an absorbing Markov chain* **P** *with* **N** *as its fundamental matrix, and* **P** *starts in state* s_i*, let* T_i *be the expected number of steps before the chain is absorbed. Let* **T** *be the column vector whose* i^{th} *entry is* T_i*. Then*

$$\mathbf{T} = \mathbf{N} \cdot \mathbf{1} \tag{6}$$

where **1** *is the column vector whose entries are all 1.*

By the previous theorem, the expected time until the chain is absorbed given the starting state s_i is given by the sum of the entries in the i_{th} row of the fundamental matrix **N**. This quantity is precisely the expected time until ruin, or average number of games played until a player's wealth is reduced to zero, given an initial nonterminal state for the three-player gambler's ruin.

The following theorem tells us how absorption probabilities are obtained using the fundamental matrix of an absorbing Markov chain [10, 12].

Theorem 3 *Let* b_{ij} *be the probability that an absorbing chain will be absorbed in the absorbing state* s_j *if it starts in the transient state* s_i*. Let* **B** *be the matrix with entries* b_{ij}*. Then* **B** *is a* $t \times r$ *matrix, and*

$$\mathbf{B} = \mathbf{NR}, \tag{7}$$

where **N** *is the fundamental matrix and* **R** *is as in the canonical form.*

RESULTS AND DISCUSSION

In this section, the connections from a nonterminal state to a terminal or nonterminal state are presented using theorems. Multigraphs and solutions for some chip states are presented. Solutions obtained from this research are compared to those from the ThreeTower scenario. Expected time until ruin for some states and wealth totals are discussed.

Multigraph Theorems

The following theorem describes the connections between a nonterminal state (x,y,z) and some terminal states.

Theorem 4 *Given a nonterminal state* (x,y,z)*, then it is connected to the following terminal states, with the indicated edge weights:*

(i) $(x+z,y,0)$, *with edge weight* $\frac{1}{6z}$

(ii) $(x,y+z,0)$ *or* $(y+z,x,0)$, *with edge weight* $\frac{1}{6z}$

(iii) $(x+y,z,0)$, *with edge weight* $\frac{1}{6y}$

Proof: We prove (i). Let (x,y,z) be a nonterminal state. Then from Definition 6, $(x,y,z) \to (x+b,y,z-b)$ with probability $\frac{1}{6}$ for some $b \in \{1,2,...,z\}$. Since b is uniformly distributed, then $b = z$ with probability $\frac{1}{z}$. So $(x,y,z) \to (x+z,y,0)$ with probability $\frac{1}{6} \cdot \frac{1}{z} = \frac{1}{6z}$. The rest of the proof follows similarly.

The following theorem describes the connections between a nonterminal state (x,y,z) and some other nonterminal states.

Theorem 5 *Given a nonterminal state* (x,y,z)*. Let* $a = \min\{y,z\}$*,* $b = \min\{x,z\}$ *and* $c = \min\{x,y\}$ *be the maximum permissible bet size for each pair of players. Then* (x,y,z) *is connected to the following nonterminal states, with the indicated edge weights:*

(i) $(x,y+n,z-n)$ *with edge weight* $\frac{1}{6a}$, *where* $n \in \{1,2,...,a\}$

(ii) $(x,y-n,z+n)$ *with edge weight* $\frac{1}{6a}$, *where* $n \in \{1,2,...,a\}$

(iii) $(x+n,y,z-n)$ *with edge weight* $\frac{1}{6b}$, *where* $n \in \{1,2,...,b\}$

(iv) $(x-n,y,z+n)$ *with edge weight* $\frac{1}{6b}$, *where* $n \in \{1,2,...,b\}$

(v) $(x+n,y-n,z)$ *with edge weight* $\frac{1}{6c}$, *where* $n \in \{1,2,...,c\}$

(vi) $(x-n,y+n,z)$ *with edge weight* $\frac{1}{6c}$, *where* $n \in \{1,2,...,c\}$

provided that each chip position in the new chip state is nonzero. The chip positions are rearranged in decreasing order to get a valid chip state.

Proof: We prove (i). Let (x,y,z) be a nonterminal state, and $a = \min\{y,z\}$. Then from Definition 6, $(x,y,z) \to (x,y+k,z-k)$ with probability $\frac{1}{6}$ for some $k \in \{1,2,...,a\}$. Since k is uniformly distributed, then $k = n$ with probability $\frac{1}{a}$. So $(x,y,z) \to (x,y+n,z-n)$ with probability $\frac{1}{6} \cdot \frac{1}{a} = \frac{1}{6a}$. The rest of the proof follows similarly.

Three-Player Game for $S = 6$

Example 1 *In this example, we take a more detailed look at the No Limit three-player gambler's ruin results for* $S = 6$*. We first look at the graph for the three-player game (Unit Bet and No Limit variants) with total wealth* $S = 6$*, as shown in Figures 1 and 2.*

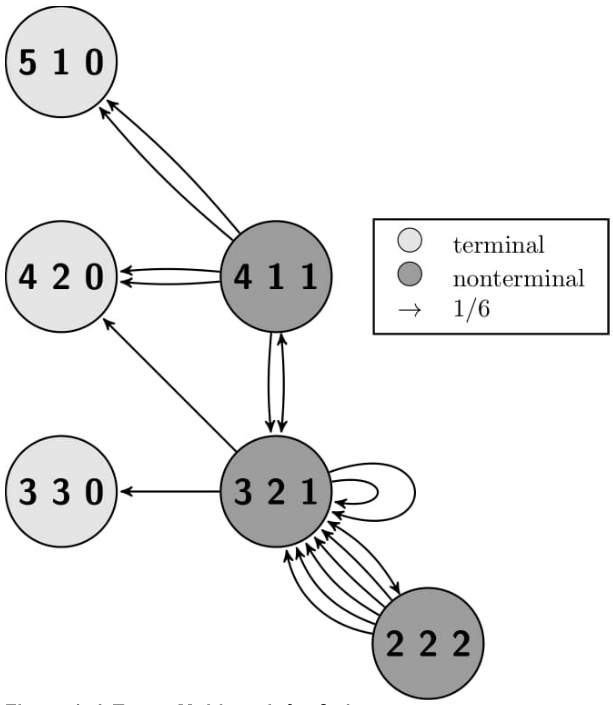

Figure 1: 3-Tower Multigraph for S=6

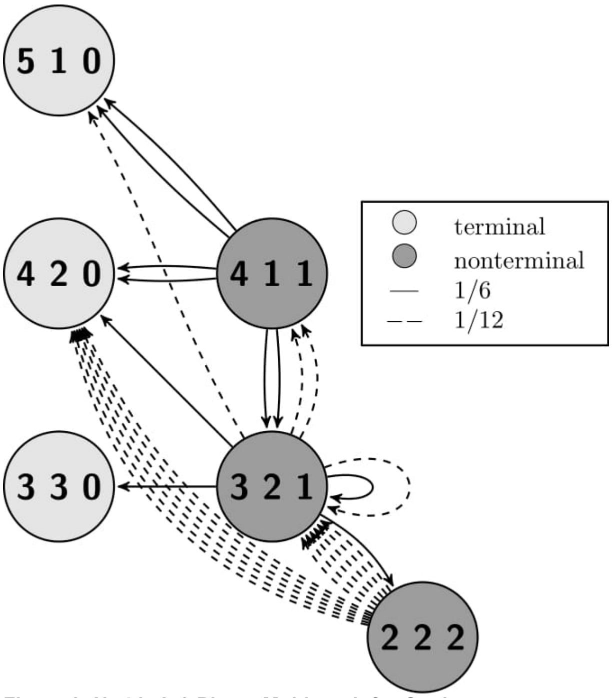

Figure 2: No Limit 3-Player Multigraph for *S* = 6

In this example, the unique terminal states are (5,1,0), (4,2,0) *and* (3,3,0)*, denoted by* P_1*,* P_2*, and* P_3*, respectively. The unique nonterminal states are* (4,1,1), (3,2,1) *and* (2,2,2)*, denoted by* P_4*,* P_5*, and* P_6*, respectively. From these states, we observe that there are six unique chip positions in nonterminal states and eight unique chip positions in terminal states. However, note that in Figure 1, all edges have weight* 1/6*, while in Figure 2, edges have weights* 1/6 *or* 1/12*. This difference in edge weights occurs due to the different permissible bet sizes in each scenario.*

In Figure 2, note that there are two outward edges from P_4 *going to* P_1*. One edge represents Player 1 beating Player 2 for 1 chip, while another edge represents Player 1 beating Player 3. All of the outward edges from* P_4 *have weight* 1/6 *since the maximum permissible bet size for each pair of players is 1. However, for* P_5*, not all outward edges have the same weight. Some edges have weight* 1/12 *since these represent the cases where Players 1 and 2 are involved in the round of betting, with the maximum permissible bet size equal to 2 instead of 1. Loops for* P_5 *are obtained since it can transition to* (3,1,2) *(with weight* 1/6*) if Player 3 beats Player 2 for 1 chip, or to* (2,3,1) *(with weight* 1/12*) if Player 2 beats Player 1 for 2 chips. Note that while* P_6 *is a nonterminal state by definition, placing probabilities for each player in this state can be computed trivially, with each player having 1/3 probability of finishing first, second or third.*

After the construction of a multigraph for a given wealth total S, a linear system representing the transitions between the states is constructed, as detailed in Section 2.2. For example, if $S = 6$*, we define six variables* v_1 *up to* v_6 *corresponding respectively to the unique nonterminal chip positions 4 and 1 in* P_4*, positions 3, 2 and 1 in* P_5*, and position 2 in* P_6*. We also define eight variables* w_1 *up to* w_8 *for terminal states, corresponding respectively to the chip positions 5, 1, and 0 in* P_1*, positions 4, 2, and 0 in* P_2*, and positions 3 and 0 in* P_3*. We get the linear system by setting up the recurrence:*

$$\left(\boldsymbol{I_6} - \frac{1}{12}\begin{bmatrix} 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 & 0 \\ 1 & 1 & 2 & 1 & 0 & 2 \\ 1 & 1 & 1 & 0 & 2 & 2 \\ 0 & 2 & 0 & 2 & 1 & 2 \\ 0 & 0 & 2 & 2 & 2 & 0 \end{bmatrix}\right)\boldsymbol{B} = \frac{1}{12}\begin{bmatrix} 4 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 2 & 2 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 2 & 2 & 2 & 0 & 0 \end{bmatrix} \quad (8)$$

The solution to this system is an $n \times m$ *matrix that gives the absorption probabilities for the* n *unique nonterminal chip positions into the* m *unique terminal chip positions. This system has the following solution:*

$$\mathbf{B} = \begin{bmatrix} 0.3843 & 0.0112 & 0.0136 & 0.4362 & 0.0343 & 0.0295 & 0.0847 & 0.0062 \\ 0.0124 & 0.1989 & 0.1977 & 0.0319 & 0.2329 & 0.2353 & 0.0486 & 0.0423 \\ 0.1528 & 0.0336 & 0.0409 & 0.3087 & 0.1029 & 0.0884 & 0.2540 & 0.0187 \\ 0.0553 & 0.0505 & 0.1215 & 0.1185 & 0.2617 & 0.1197 & 0.2251 & 0.0476 \\ 0.0192 & 0.1431 & 0.0649 & 0.0728 & 0.1354 & 0.2918 & 0.0663 & 0.2064 \\ 0.0379 & 0.0379 & 0.0379 & 0.2500 & 0.2500 & 0.2500 & 0.0909 & 0.0455 \end{bmatrix} \quad (9)$$

which means that for v_1 *(4 in* P_4*), its probability of ending up in the terminal chip position* w_1 *(5 in* P_1*) is* 38.43%*, and so on.*

Matrix **B** *gives the absorption probabilities given an initial position* v_i *and final position* w_j *for* $1 \leq i \leq 6$ *and* $1 \leq j \leq 8$*. To calculate the players' placing probabilities, we need the solutions to the two-player game for* $S = 6$*. In this scenario, the players' winning probabilities are equal to the proportion of their wealths to the total wealth of all players. We have the following matrix* **W** *whose entries are the placing probabilities for each nonterminal chip position in the three-player game.*

$$W = \begin{array}{c} \\ 5 \\ 1 \\ 0 \\ 4 \\ 2 \\ 0 \\ 3 \\ 0 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 5/6 & 1/6 & 0 \\ 1/6 & 5/6 & 0 \\ 0 & 0 & 1 \\ 2/3 & 1/3 & 0 \\ 1/3 & 2/3 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \quad (10)$$

Using **B** *in Equation 9 and* **W** *in Equation 10, we get matrix* **BW***, whose entries are the placing probabilities for each nonterminal chip position in the three-player game. This matrix is shown in Equation 11:*

$$\mathbf{BW} = \begin{array}{c} \\ 4 \\ 1 \\ 3 \\ 2 \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.6667 & 0.2840 & 0.0493 \\ 0.1667 & 0.3580 & 0.4753 \\ 0.5000 & 0.3520 & 0.1480 \\ 0.3333 & 0.3779 & 0.2888 \\ 0.1667 & 0.2702 & 0.5632 \\ 0.3333 & 0.3333 & 0.3333 \end{bmatrix} \end{array} \quad (11)$$

For comparison, matrix **C** *in Equation 12 is the matrix containing placing probabilities for each nonterminal chip position in the Three-Tower game [12].*

$$\mathbf{C} = \begin{array}{c} \\ 4 \\ 1 \\ 3 \\ 2 \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.6667 & 0.2910 & 0.0423 \\ 0.1667 & 0.3545 & 0.4788 \\ 0.5000 & 0.3731 & 0.1269 \\ 0.3333 & 0.4078 & 0.2589 \\ 0.1667 & 0.2191 & 0.6142 \\ 0.3333 & 0.3333 & 0.3333 \end{bmatrix} \end{array} \quad (12)$$

Now, the fundamental matrix **N** *is given as follows:*

$$N = \left(I_6 - \frac{1}{12}\begin{bmatrix} 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 & 0 \\ 1 & 1 & 2 & 1 & 0 & 2 \\ 1 & 1 & 1 & 0 & 2 & 2 \\ 0 & 2 & 0 & 2 & 1 & 2 \\ 0 & 0 & 2 & 2 & 2 & 0 \end{bmatrix} \right)^{-1} = \begin{bmatrix} 197/189 & 17/350 & 91/206 & 12/181 & 11/294 & 1/11 \\ 6/247 & 1617/1516 & 7/135 & 23/96 & 79/311 & 1/11 \\ 79/622 & 29/199 & 273/206 & 36/181 & 11/98 & 3/11 \\ 60/533 & 41/256 & 16/75 & 1125/989 & 217/760 & 3/11 \\ 6/181 & 23/96 & 40/409 & 914/3047 & 353/285 & 3/11 \\ 1/22 & 1/11 & 3/11 & 3/11 & 3/11 & 25/22 \end{bmatrix} \quad (13)$$

By Theorem 2,

$$\mathbf{T} = \mathbf{N}(1\ 1\ 1\ 1\ 1\ 1)^T = (19/11\ 19/11\ 24/11\ 24/11\ 24/11\ 23/11)^T \quad (14)$$

which gives the expected time until ruin for the nonterminal states for $S = 6$. Recall that the first two rows of **N** *correspond to the unique positions of the state* $(4,1,1)$*, the next three rows for the unique positions of* $(3,2,1)$*, and the last row for the unique position in the state* $(2,2,2)$*. This means that the expected time until ruin for the states* $(4,1,1)$*,* $(3,2,1)$ *and* $(2,2,2)$ *are* $19/11, 24/11$ *and* $23/11$ *time steps, respectively.*

Expected Time until Ruin

Table 1 shows the expected time until ruin for $S = 4,5,6,7$ and 8 for the Unit Bet and No Limit three-player gambler's ruin scenarios.

Table 1: Expected Time Until Ruin for States with *S* = 4,5,6,7 and 8

		Expected Time Until Ruin	
S	State	Unit Bet	No Limit Bet
4	(2,1,1)	1.500000	1.500000
5	(3,1,1)	1.800000	1.636364
	(2,2,1)	2.400000	1.909091
6	(4,1,1)	2.000000	1.727273
	(3,2,1)	3.000000	2.181818
	(2,2,2)	4.000000	2.090909
7	(5,1,1)	2.142857	1.791045
	(4,2,1)	3.428571	2.373134
	(3,3,1)	3.857143	2.334755
	(3,2,2)	5.142857	2.616205
8	(6,1,1)	2.250000	1.817985
	(5,2,1)	3.750000	2.453954
	(4,3,1)	4.500000	2.570423
	(4,2,2)	6.000000	2.790358
	(3,3,2)	6.750000	2.927411

Expected time until ruin for the Unit Bet case is calculated using the formula $T = \frac{3ABC}{A+B+C}$ [3]. For example, for the state $(3,2,1)$, expected time until ruin is

$$T = \frac{3ABC}{A+B+C} = \frac{3 \cdot 3 \cdot 2 \cdot 1}{3+2+1} = 3 \quad (15)$$

For the No Limit case, Theorem 2 is used in calculating the expected time until ruin. An example of this calculation is shown in Section 3.2 for states with $S = 6$. Note that if we use Theorem 2 in calculating the expected time until ruin for the Unit Bet case, we get identical results as the known formula, validating our calculation method for the expected time until ruin.

Observe that for any state, the expected time until ruin for the Unit Bet case is greater than or equal to that for the No Limit case. This is true because of the difference in nature of the transfer of wealth between pairs of players in each round of the two cases. For the Unit Bet case, exactly one chip is transferred between the players involved in the round. This means that if all three players have more than one chip, it is impossible for a player to "bust out" or run out of chips in a single round. Only players with one chip at the start of the round can be ruined in that round. States of this form are called near-terminal states since they can go to a terminal state in a single step. For the No Limit case, more than one chip may be transferred between the players. In fact, all the chips of the player with the smaller wealth total may be transferred to the other player in a single round. This results in a higher probability of a player running out of chips in a single round. Hence, a player may achieve ruin faster, leading to a lower expected time until ruin. Also, if the two players involved in the round have equal wealth totals, each of them has a nonzero probability of achieving ruin in a single round.

Also, observe that the only case when the expected time until ruin for the Unit Bet and No Limit games are equal is the state $(2,1,1)$ for $S = 4$. The expected times are equal since the Unit Bet and No Limit scenarios for $S = 4$ are identical. In fact, it is the only instance where the Unit Bet and No Limit scenarios are identical. For values of S greater than 4, we observe that the expected time until ruin for the Unit Bet scenario is strictly greater than that of the No Limit scenario. This conjecture may be proved in another paper.

Table 2 shows the expected time until ruin for $S = 12$ for the Unit Bet and No Limit three-player gambler's ruin scenarios.

Table 2: Expected Time Until Ruin for States with *S* = 12

	Expected Time Until Ruin	
State	Unit Bet	No Limit Bet
(10,1,1)	2.50	1.870516
(9,2,1)	4.50	2.611548
(8,3,1)	6.00	2.890920
(7,4,1)	7.00	3.045563
(6,5,1)	7.50	3.046073
(8,2,2)	8.00	3.154241
(7,3,2)	10.50	3.651354
(6,4,2)	12.00	3.771622
(5,5,2)	12.50	3.728184
(6,3,3)	13.50	3.800937
(5,4,3)	15.00	4.023762
(4,4,4)	16.00	3.710237

Note that the states are arranged in increasing order according to the third coordinates, then the second coordinates. Observe that the expected time until ruin for the Unit Bet case is increasing as we go down the column. Recall that the expected time until ruin for a state (A,B,C) in the Unit Bet case is given by the formula $T = \frac{3ABC}{A+B+C}$ [2, 7, 13]. Clearly, expected time until ruin is maximized when the three players start with equal wealths.

Now, observe that while the expected time until ruin for the No Limit case is generally increasing as we go down the column, there are some cases when the expected time decreases. For $S=12$, let's look at two specific states: (5,5,2) and (4,4,4).

For the state (5,5,2), notice that the top two players have the same wealth total. We compare this with the previous state (6,4,2), which has three non-equal positions. For the state (5,5,2), all three players can be ruined in a single round. For example, the first and second players can be ruined if they face each other in the round, with bet size equal to 5. Player 3 can be ruined in any round facing Player 1 and 2 if he loses the round with bet size equal to 2. Calculating the probability of a player being ruined in one round, we have $2\left(\frac{1}{6}\right)\left(\frac{1}{5}\right)+2\left(\frac{1}{6}\right)\left(\frac{1}{2}\right)=\frac{7}{30}$. For the state (6,4,2), Player 1 has zero probability of being ruined in a single round since the maximum permissible bet sizes involving them is either 2 or 4. Only Players 2 and 3 can be ruined in a single round. Calculating the probability of a player being ruined in one round, we have $1\left(\frac{1}{6}\right)\left(\frac{1}{4}\right)+2\left(\frac{1}{6}\right)\left(\frac{1}{2}\right)=\frac{5}{24}$, which is smaller than $\frac{7}{30}$. Hence, it is more likely that a player is ruined in the first round for the state (5,5,2), leading to a smaller expected time until ruin for this state.

For the state (4,4,4), all three players have the same wealth total. This means that in the first round, any player may be ruined if the selected bet size is equal to 4. The probability of a player being ruined in one round is $6\left(\frac{1}{6}\right)\left(\frac{1}{4}\right)=\frac{1}{4}$. Hence, it is more likely that a player is ruined in the first round for this state compared to the state (5,5,2), leading to a smaller expected time until ruin for the state (4,4,4).

Other Results

We compare the results obtained for the placing probabilities for some states of the form $(3k,2k,k)$ using the multigraph model and the Independent Chip Model (ICM) [11], which is a widely used but simple approximation. Table 3 shows these placing probabilities. First place probabilities are the same for both models. We observe that the solutions obtained from the multigraph model depend on the actual number of chips of each player. Note that placing probabilities using the ICM depend only on the proportion of the chip stacks, which means that all states of the form $(3k,2k,k)$ where $k\in\mathbb{N}$ have the same solution using the ICM. These subtle differences in placing probabilities for states having a constant chip ratio would have been difficult to detect using other models involving simulations. For a discussion on using regression with ICM to get good approximations for ruin probabilities, see Diaconis and Ethier (2020) [6].

Table 3: Solutions for Some States of the Form (3*k*,2*k*,*k*)

State	Position	$P(X_i=1)$	$P(X_i=2)$	$P(X_i=3)$
(3,2,1)	3	0.500000	0.351986	0.148014
(6,4,2)	6	0.500000	0.345394	0.154606
(12,8,4)	12	0.500000	0.338341	0.161659
(24,16,8)	24	0.500000	0.331857	0.168143
(48,32,16)	48	0.500000	0.326231	0.173769
(96,64,32)	96	0.500000	0.321519	0.178481
ICM	$3k$	0.500000	0.350000	0.150000
(3,2,1)	2	0.333333	0.377858	0.288809
(6,4,2)	4	0.333333	0.372622	0.294044
(12,8,4)	8	0.333333	0.368471	0.298195
(24,16,8)	16	0.333333	0.364867	0.301799
(48,32,16)	32	0.333333	0.361926	0.304740
(96,64,32)	64	0.333333	0.359609	0.307057
ICM	$2k$	0.333333	0.400000	0.266667
(3,2,1)	1	0.166667	0.270156	0.563177
(6,4,2)	2	0.166667	0.281984	0.551349
(12,8,4)	4	0.166667	0.293188	0.540145
(24,16,8)	8	0.166667	0.303276	0.530057
(48,32,16)	16	0.166667	0.311843	0.521491
(96,64,32)	32	0.166667	0.318872	0.514462
ICM	k	0.166667	0.250000	0.583333

For some wealth totals S, Table 4 shows the run time (in seconds) of our program in solving for placing probabilities and the number of operations needed in inverting matrix $\mathbf{I}-\mathbf{Q}$. Diaconis and Ethier [6] discussed a Markov chain model where states are represented as interior and boundary points in a triangular lattice. They noted that the only computationally difficult part of their program was inverting an $\binom{S-1}{2}\times\binom{S-1}{2}$ matrix. In the multigraph model, matrix size is equal to the number of unique nonterminal chip positions for the given wealth total. It is known that the inversion of an $n\times n$ matrix takes $\frac{1}{3}(n^3-n)=\mathcal{O}(n^3)$ operations. Since the multigraph model uses a smaller matrix which has size about half of that of the triangular lattice model, it yields considerable savings in the number of operations. When $S=321$ (the largest S for which we have results), matrix size is 25,600 × 25,600 and program runtime was about 43 minutes in a laptop with an AMD Ryzen 7 2.00 GHz processor with 8 cores and 8 GB RAM.

Table 4: Run Time and Number of Operations for Matrix Inversion for Some Values of *S*

	Lattice Model		Multigraph Model			
S	n	No. of Operations	n	No. of Operations	Run time (s)	Savings
6	10	330	6	70	0.01	78.8%
12	55	55440	30	8990	0.03	83.8%
24	253	5398008	132	766612	0.14	85.8%
48	1081	4.21×10^{8}	552	5.61×10^{7}	1.12	86.7%
96	4465	2.97×10^{10}	2256	3.83×10^{9}	16.00	87.1%
192	18145	1.99×10^{12}	9120	2.53×10^{11}	271.33	87.3%

Although considerable savings in run time is presented, the multigraph model is limited in other ways. For example, elimination-order probabilities cannot be solved directly without making significant changes to the algorithm. It is also limited to studies involving fair games, where any player has an equal chance of winning against each of the other players. These limitations stem from the treatment of states (a,b,c), (a,c,b), (b,a,c), (b,c,a), (c,a,b) and (c,b,a) as the same state since we focused on solving for placing probabilities in this study.

CONCLUSIONS

In this paper, a method of solving player equities in the No Limit three-player gambler's ruin was presented. The assumptions for the problem were that betting was even-money with no draw, bet size may vary depending on the wealth of the pair of players involved in each round, and the participating players and bet sizes for each round were selected randomly following a uniform distribution. The method used recursions with players having integer wealths at the beginning of rounds. A multigraph with nodes representing the different states for a fixed total wealth S was constructed, and a linear system was constructed to represent the transition between these states. Solutions of this linear system give the absorption probabilities for each player in all possible states and placing probabilities can be obtained by using known placing probabilities for the two-player game. Expected time until ruin can also be solved from the constructed linear system, but a formula has not been obtained for this expectation. Considerable savings in run time is presented when comparing the multigraph model with the standard triangular lattice model.

The model may be extended to one wherein the selection of bet size follows a distribution other than the uniform distribution. This may be a better approximation of most poker tournaments since bet size between players usually belongs to the lower range of permissible bet sizes. The model may also be applied to other variations of the three-player game such at the player-centric and symmetric games. The model may also be extended to a general *N*-player problem, which the authors plan to do.

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CONFLICTS OF INTEREST

The authors declare no competing interests.

CONTRIBUTIONS OF INDIVIDUAL AUTHORS

R.I.D. Marfil designed the model, analyzed the data, authored drafts of the paper, and approved the final draft.

G. David approved the proposed model, reviewed drafts of the paper, and approved the final draft.

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