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Fragments that reduce the gap cap of the T-tetromino to six

Vince Jan F. Torres and Louie John D. Vallejo*

Institute of Mathematics, College of Science, University of the Philippines Diliman, Quezon City, Philippines

ABSTRACT

In 2015, it was established that at least nine monominoes are needed to tile all sufficiently large rectangles by T-tetrominoes and monominoes. In this paper, we present tiling schemes that use special corner fragments to reduce this upper bound to six.

INTRODUCTION

A T-tetromino is a plane figure consisting of four unit squares that are joined edge-to-edge to form a T. It is one of five tetrominoes along with the L-, I- or straight, O- or square, and S- or skew tetrominoes that many readers would recognize as the tile pieces used in the game *Tetris*. They form part of a larger class of plane figures called polyominoes which were popularized in the 1950s by S. Golomb. A standard reference is (Golomb 1994) which synthesized several research articles on polyominoes. Another good puzzle and problem book about polyominoes is (Martin 1996).

Many research questions about polyominoes are about constrained tiling of plane regions such as rectangles. A complete tiling of a rectangle by T-tetrominoes requires filling out the whole region of the rectangle with only T-tetrominoes in any of their four possible orientations (see Figure 1) such that the tiles do not overlap except possibly at their corners or edges.

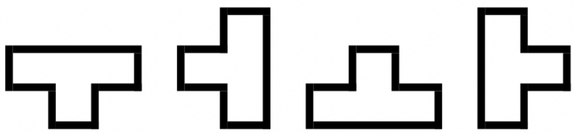

Figure 1: The four orientations of a T-tetromino.

In 1965, Walkup proved that a $w \times n$ rectangle can be completely tiled by T-tetrominoes alone if and only if w and n are both divisible by 4 (see Figure 2). The identification of distinct complete tilings of $4a \times 4c$ rectangles by T-tetrominoes is the subject of (Pak&Korn 2004) and (Feller&Hochberg 2024).

*Corresponding author
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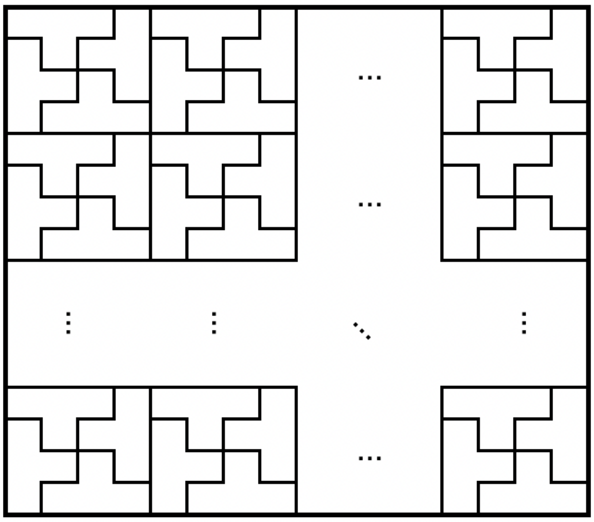


Figure 2: A complete tiling of a $4a \times 4c$ rectangle by T-tetrominoes.

If a rectangle $\boldsymbol{R}$ does not satisfy the condition of Walkup, then tiling $\boldsymbol{R}$ with T-tetrominoes will necessarily leave gaps. A complete tiling of $\boldsymbol{R}$ can still be done if we allow monomino tiles (a unit square tile) to fill these gaps. A natural question then is to find the minimum number of monominoes required to completely tile a rectangle by only using T-tetrominoes and monominoes. This was pursued by (Zhan 2012) and more comprehensively by (Hochberg 2015). Their pertinent results are enumerated in Theorem 1.

Let us denote by $M(w,n)$ the least number of monominoes needed to tile a $w \times n$ rectangle by T-tetrominoes and monominoes. For a fixed width w, the sequence $\{M(w,n)\}_n$ can blow up. For example, it is trivial to see that $M(1,n) = n \to +\infty$. In fact, if w is a fixed odd integer between 1 and 12, then $M(w,n) \to +\infty$ (Hochberg 2015). Theorem $1(iii)$ below is therefore a remarkable result. We enumerate below the state of the art about the quantity $M(w,n)$.

Theorem 1: Let w and n be positive integers. Then

- *(i)* $M(w,n) = 0$ if and only if $w \equiv n \equiv 0(mod 4)$ (Walkup 1965)
- *(ii)* $M(w,n) = 1$ if and only if $w = n = 1$ (Zhan 2012)
- *(iii)* $M(w,n) \leq 9$ for all integers $w \geq 12$ and $n \geq 12$ (Hochberg 2015)
- *(iv)* $M(n,n) \leq 5$ for all positive integers n (Grahl 2018).

Hochberg coined the term *Gap Cap* to denote

$$G = \limsup_{\min(w,n)\to+\infty} M(w,n). \qquad (1)$$

With this notation, Theorem $1(iii)$ is equivalent to $G \leq 9$. Hochberg however conjectured that $G = 5$. This is supported by Grahl who constructed optimal complete tilings of squares of any size that use at most five monominoes. We recover Grahl's result for large squares in Lemma 5 and Lemma 7, and improve Hochberg's result in Theorem 8, where we prove that $G \leq 6$.

THE GAP CAP IS BETWEEN 5 AND 9

Because T-tetrominoes consist of four unit squares, the area of a $w \times n$ rectangle that is completely tiled with a T-tetrominoes and c monominoes is given by $wn = 4a + c$. Therefore,

$$M(w,n) \equiv wn \ (mod\ 4). \qquad (2)$$

This gives rise to the following expected result.

Proposition 2: $G \geq 5$.

Proof: In view of (2), we get that for all nonnegative integers a and c,

$$M(4a+1, 4c+1) \equiv 1 \ (mod\ 4). \qquad (3)$$

However, Theorem $1(ii)$ asserts that the left-hand side of (3) is at least 5, if a or c is not zero. This proves the proposition. ■

Next, we illustrate how Hochberg proved that $G \leq 9$. The important ingredient to his proof is his discovery of 13- and 15-width T-tetromino cylinders which are fixed-width fragments that are completely tiled by T-tetrominoes. Moreover, their left and right boundaries fit with each other so that they can periodically tile an infinite strip of a certain width like a frieze pattern. In Figure 3, we present a variant of the cylinders found in (Figure 12, Hochberg 2015) where we have displaced some tiles from one boundary to another with the objective of minimizing the number of non-full columns. Note that if the jagged boundary of each of these cylinders are connected, each will produce a cylindrical surface that spans 16 full columns. The 13-width strip is the smallest odd-width strip that can be completely tiled by T-tetrominoes alone (Hochberg 2015).

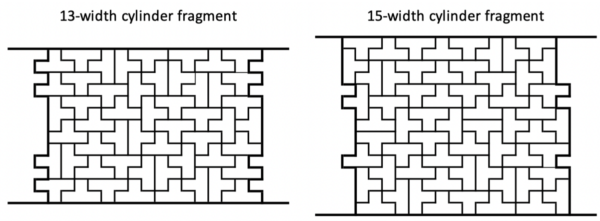
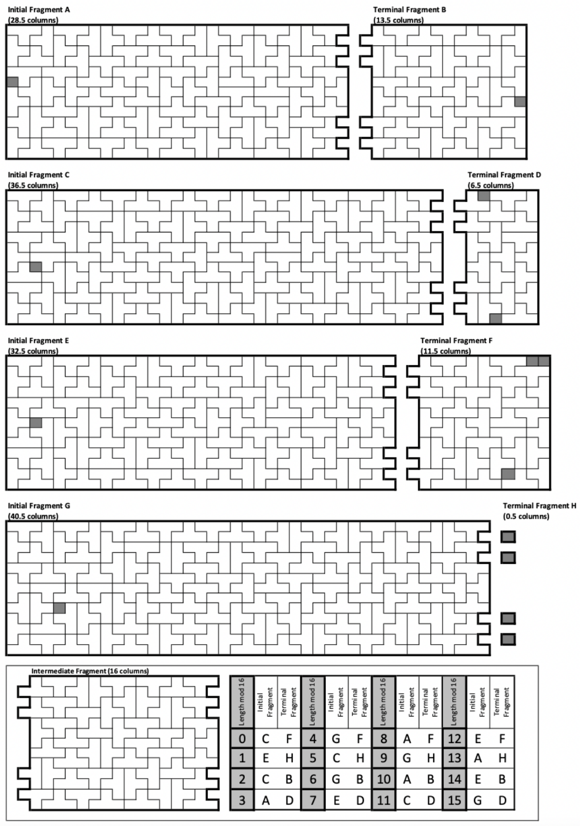


Figure 3: T-tetromino cylinders of widths **13** (left) and **15** (right).

These T-tetromino cylinders allow us to control the number of monominoes in a tiling of a $\mathbf{13} \times \boldsymbol{n}$ or a $\mathbf{15} \times \boldsymbol{n}$ rectangle for large $\boldsymbol{n}$. We only need to find initial and terminal fragments that conform to the cylinders' jagged boundary and insert as many of these T-tetromino cylinders as needed. Since the cylinders do not need any monominoes, we can then just minimize the number of monominoes in the tiling of these initial and terminal fragments.

In Figure 4, we present four initial fragments and four terminal fragments for 13-width rectangles. These are variants of those found in (Figure 13, Hochberg 2015). In each fragment, the number of full columns is counted and any non-full column is given a count of 0.5. This is consistent with the fact that the boundaries fit with each other so that connecting two fragments or a fragment and the 13-width cylinder along their jagged boundaries adds another full column.

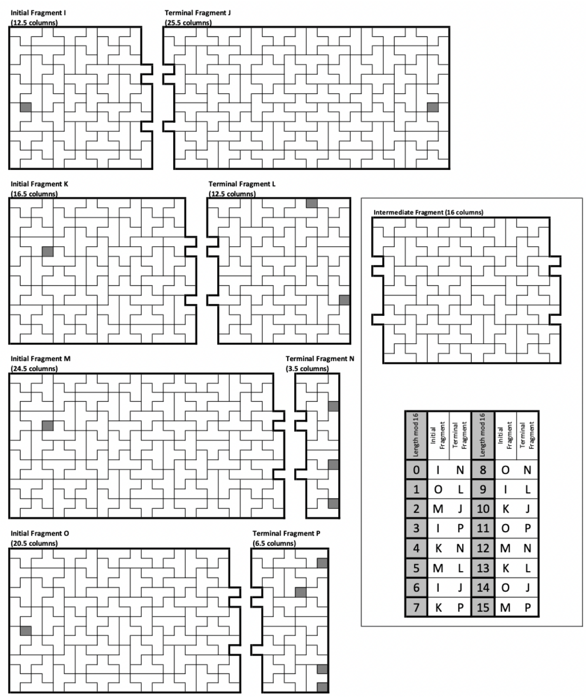
It is easy to check that the 16 combinations of initial-terminal fragment pairs have distinct lengths modulo 16. Since our 13-width cylinder fragment in Figure 3 forms 16 full columns (actually, 15 full columns plus 2 non-full columns), we are able to tile any rectangle of size $\mathbf{13} \times \boldsymbol{n}$ with $\boldsymbol{n} \geq \mathbf{39}$. For example, to tile the $\mathbf{13} \times \mathbf{100}$ rectangle, we first observe that $\mathbf{100} \equiv \mathbf{4}\ (\boldsymbol{mod}\ \mathbf{16})$. The reference table at the bottom of Figure 4 directs us to choose the (G,F) initial-terminal fragment pair which already gives 52 full columns. To make up for the 48 more columns, we insert three 13-width cylinders between the initial fragment G and terminal fragment F.



Length mod 16	Initial Fragment	Terminal Fragment	Length mod 16	Initial Fragment	Terminal Fragment	Length mod 16	Initial Fragment	Terminal Fragment	Length mod 16	Initial Fragment	Terminal Fragment
0	C	F	4	G	F	8	A	F	12	E	F
1	E	H	5	C	H	9	G	H	13	A	H
2	C	B	6	G	B	10	A	B	14	E	B
3	A	D	7	E	D	11	C	D	15	G	D

Figure 4: Initial and terminal fragments needed to tile $\mathbf{13 \times n}$ rectangles. The reference table at the end of the figure provides the appropriate choice of initial-terminal fragment pairs.

Similarly, we present in Figure 5 the four initial fragments and four terminal fragments needed to tile $15 \times n$ rectangles, with $n \geq 35$. These are variants of those found in (Figure 15, Hochberg 2015). For example, the 15×35 rectangle can be tiled by linking Fragment I, the 15-width cylinder, and Fragment P, while the 15×36 rectangle can be tiled by linking Fragment K, the 15-width cylinder, and Fragment N.



Length mod 16	Initial Fragment	Terminal Fragment	Length mod 16	Initial Fragment	Terminal Fragment
0	I	N	8	O	N
1	O	L	9	I	L
2	M	J	10	K	J
3	I	P	11	O	P
4	K	N	12	M	N
5	M	L	13	K	L
6	I	J	14	O	J
7	K	P	15	M	P

Figure 5: Initial and terminal fragments needed to tile $\mathbf{15 \times n}$ rectangles. The reference table at the bottom right of the figure provides the appropriate choice of initial-terminal fragment pairs.

We are now ready to completely tile large rectangles with T-tetrominoes and monominoes. For example, a large $(4j+1) \times (4k+3)$ rectangle can be subdivided into a $4a \times 4c$ rectangle, a $13 \times (4c+15)$ rectangle and a $4a \times 15$ rectangle (see Figure 6). The first $4a \times 4c$ rectangle can be completely tiled by T-tetrominoes alone in view of Theorem 1(i). The next $13 \times (4c+15)$ rectangle can be tiled with initial-terminal fragment pair (A,D), (E,D), (C,D), or (G,D) (with labels found in Figure 4) if $c \equiv 1,2,3$, or $0\ (mod\ 4)$, respectively. In any case, a tiling of this rectangle by T-tetrominoes and monominoes would require at least three monominoes. Finally, the $4a \times 15$ rectangle can be tiled with initial-terminal fragment pair (K,N), (O,N), (M,N), or (I,N) in Figure 5 depending on the value of a. In any case, this would require at least four monominoes.

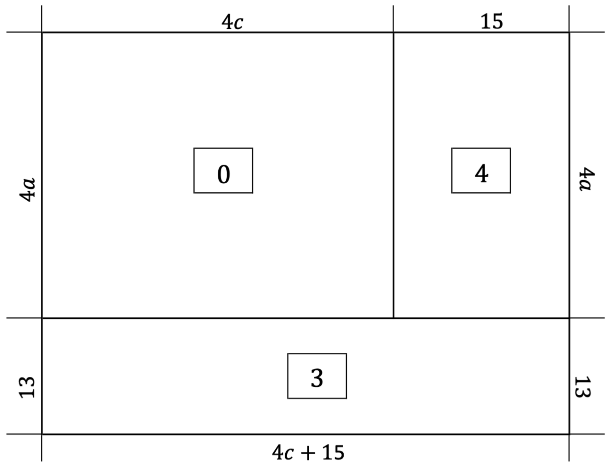


Figure 6: Decomposition of a large $\mathbf{(4j+1) \times (4k+3)}$ rectangle. The numbers inside the rectangular subregions correspond to the number of monominoes needed in tiling each subregion according to the scheme described in Figures 4 and 5.

The following theorem summarizes all the information found in (Figure 16, Hochberg 2015). Aside from the configurations described in Figures 4 and 5, the proof also uses the easily verifiable fact that

$$M(2,2n) = 4 \text{ for all positive integers } n. \qquad (4)$$

Theorem 3 (Hochberg 2015): For all integers $j \geq 3$ and $k \geq 3$,

(i) $M(4j, 4k+1) = M(4a, 4c+13) = 4$
(ii) $M(4j, 4k+2) = 4$
(iii) $M(4j, 4k+3) = M(4a, 4c+15) = 4$
(iv) $M(4j+1, 4k+1) = M(4a+13, 4c+13) \leq 9$
(v) $M(4j+1, 4k+2) = M(4a+13, 4c+2) \leq 6$
(vi) $M(4j+1, 4k+3) = M(4a+13, 4c+15) \leq 7$
(vii) $M(4j+2, 4k+2) = 4$
(vii) $M(4j+2, 4k+3) = M(4a+2, 4c+15) \leq 6$
(ix) $M(4j+3, 4k+3) = M(4a+15, 4c+15) \leq 9$

Consequently, along with Walkup's Theorem, we obtain $G \leq 9$.

Remark 4:

1. Although the tiling schemes depicted in Figures 4 and 5 work only for $13 \times n$ and $15 \times m$ rectangles, with $n \geq 39$ and $m \geq 35$, Theorem 7 in (Hochberg 2015) asserts that the same conclusion $G \leq 9$ still holds even for $12 \leq n \leq 38$ and $12 \leq m \leq 34$, where in some cases, computer-aided search was implemented.
2. The statements in (i), (ii), (iii), and (vii) are equalities because the quantities cannot be made smaller in view of Theorem 1(i).
3. To see (vii), the L-shaped region found by deleting a corner $4j \times 4k$ rectangle from the $(4j+2) \times (4k+2)$ rectangle can be completely tiled by T-tetrominoes and exactly 4 monominoes (see Figure 7). Another optimal

tiling for $(4j+2)\times(4j+2)$ squares can be found in (Proposition 2.3, Grahl 2018).

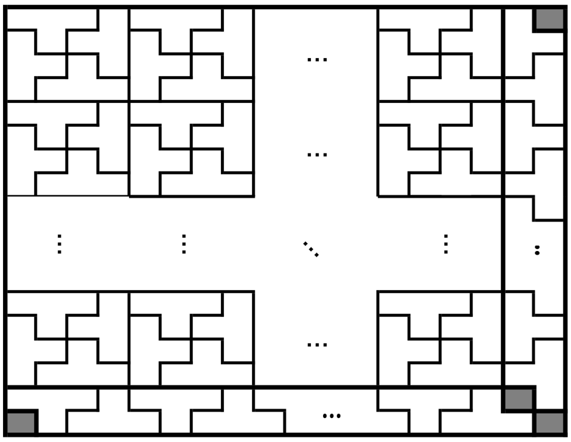


Figure 7: An optimal tiling of a $(\mathbf{4}\boldsymbol{j}+\mathbf{2})\times(\mathbf{4}\boldsymbol{k}+\mathbf{2})$ rectangle by T-tetrominoes and four monominoes.

MAIN RESULTS: TURNING THE CORNER

To achieve our $G \leq 6$ goal, it is enough to work on Statements (iv), (vi), and (ix) in Theorem 3 and propose tiling schemes to show that for all sufficiently large j and k, $M(4j+1,4k+1)=5$, $M(4j+1,4k+3)=3$, and $M(4j+3,4k+3)=5$. The prescription of these values is consistent with Equation (2) and the fact that 3 or 5 monominoes cannot be made smaller.

In his concluding discussion, Hochberg stated that, for instance in Figure 6, his division of the rectangle into three regions might not have resulted in an optimal tiling in some cases. He conjectured that it is possible to decrease this upper bound of 9 by cleverly "turning the corner" with a corner piece.

This is illustrated in Figure 7. If we had divided the $(4j+2)\times(4k+2)$ rectangle into a $4j\times 4k$ rectangle, a $2\times(4k+2)$, and a $4j\times 2$ rectangle, we would have needed at least 8 monominoes in view of Equation (4). We achieved the optimal number of 4 monominoes because we have given attention to the optimal tiling of the bottom-right corner rather than relying upon the usual tiling of a $2\times 2n$ rectangle.

The technique of "turning the corner" therefore necessitates producing special corner fragments that would eliminate the need for terminal fragments. Since all initial fragments in Figures 4 and 5 require only one monomino and the terminal fragments require more, we achieve a decrease in the Gap Cap.

Lemma 5: $M(4j+1,4k+1)=5$ for all integers $j\geq 11$ and $k\geq 11$.

Proof: We introduce the (13,13)-Corner Fragment in Figure 8.

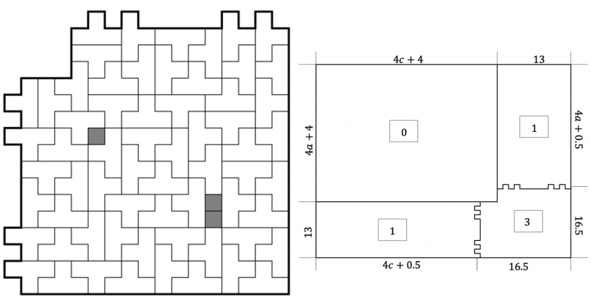


Figure 8: (Left) The corner fragment that serves as the terminal fragment for either initial fragments labeled A, C, E, or G in Figure 4 or the 13-width T-tetromino cylinder. (Right) Subdivision of a $(\mathbf{4}\boldsymbol{j}+\mathbf{1})\times(\mathbf{4}\boldsymbol{k}+\mathbf{1})$ rectangle into a rectangle, two 13-width fragments and a corner fragment.

We subdivide the given rectangle into four pieces: a $(4a+4)\times(4c+4)$ rectangle, a $13\times(4c+0.5)$ fragment, a $(4a+0.5)\times 13$ fragment, and the remaining corner fragment. This division is illustrated at the right-hand side of Figure 8. Note that in Figure 4, the 13-width initial fragment with the shortest length is Fragment A, which when connected to the (13,13)-Corner Fragment gives 45 full columns. This tells us that the smallest $(4j+1)\times(4k+1)$ rectangle that can be tiled using the scheme depicted in Figure 8 is a 45×45 rectangle.

The $(4a+4)\times(4c+4)$ rectangle needs no monominoes by Theorem 1(i). The corner fragment will be tiled as illustrated in the tiling of the (13,13)-Corner Fragment in Figure 8. This needs three monominoes. If $a\geq 7$ and $c\geq 7$, the two 13-width fragments may be tiled according to the scheme in Figure 4. Depending on the value of a, the $13\times(4a+0.5)$ fragment uses either initial fragments A, C, E, or G in Figure 4 with possible extensions by 13-width T-tetromino cylinders. This would need only one monomino. Similarly, the $(4c+0.5)\times 13$ fragment would also need one monomino. In total, this tiling scheme of the whole rectangle needs only five monominoes, which is what we wanted to show. ■

Lemma 6: $M(4j+3,4k+1)=3$ for all integers $j\geq 11$ and $k\geq 6$.

Proof: We introduce the (13,15)-Corner Fragment in Figure 9.

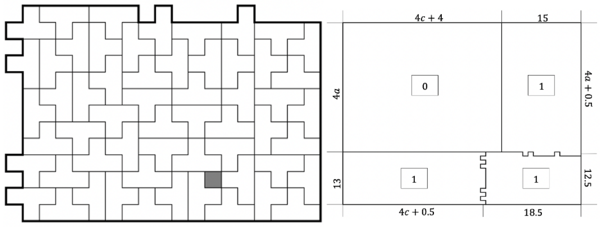


Figure 9: (Left) The corner fragment that serves as the horizontally-oriented terminal fragment for either initial fragments labeled A, C, E, or G in Figure 4 or the 13-width T-tetromino cylinder, and also serves as the vertically-oriented terminal fragment for either flipped initial fragments labeled I, K, M, or O in Figure 5 or the flipped 15-width T-tetromino cylinder. (Right) Subdivision of a $(\mathbf{4}\boldsymbol{j}+\mathbf{1})\times(\mathbf{4}\boldsymbol{k}+\mathbf{3})$ rectangle into a rectangle, a 13-width fragment, a 15-width fragment, and a corner fragment.

We subdivide the given rectangle into four pieces: a $4a\times(4c+4)$ rectangle, a $13\times(4c+0.5)$ fragment, a $(4a+0.5)\times 15$ fragment and the remaining corner fragment. This division is illustrated at the right-hand side of Figure 9. Note that in Figure 5, the 15-width initial fragment with the shortest length is Fragment I, which when flipped and connected to the (13,15)-Corner Fragment in the vertical orientation gives 25 full rows. This forces $k\geq 6$. On the other hand, connecting the smallest 13-width initial Fragment A in Figure 4 to the (13,15)-Corner Fragment gives 47 full columns. This forces $j\geq 11$. Therefore, the smallest $(4j+3)\times(4k+1)$ rectangle that can be tiled using the scheme depicted in Figure 9 is a 47×25 rectangle.

The $4a\times(4c+4)$ rectangle needs no monominoes by Theorem 1(i). The corner fragment will be tiled as illustrated in the tiling of the (13,15)-Corner Fragment in Figure 9. This needs only one monomino. Depending on the value of c, the $13\times(4c+0.5)$ fragment uses either initial fragments labeled A, C, E, or G in Figure 4 with possible extensions by 13-width T-tetromino cylinders. This would need only one monomino.

Depending on the value of a, the $(4a+0.5)\times 15$ fragment uses either flipped initial fragments labeled I, K, M, or O in Figure 5 with possible extensions by flipped 15-width T-tetromino cylinders. Note that flipping the fragments is necessary to fit the corner fragment. This would also need only one monomino. In total, this tiling scheme of the whole rectangle needs only three monominoes, which is what we wanted to show. ■

Lemma 7: $M(4j+3,4k+3)=5$ for all integers $j\geq 6$ and $k\geq 6$.

Proof: We introduce the $(15,15)$-Corner Fragment in Figure 10.

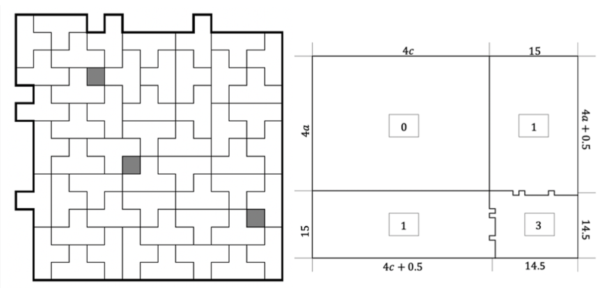


Figure 10: (Left) The corner fragment that serves as the terminal fragment for either initial fragments labeled I, K, M, or O in Figure 5 or the 15-width T-tetromino cylinder. (Right) Subdivision of a $(4j+3)\times(4k+3)$ rectangle into a rectangle, two 15-width fragments and a corner fragment.

We subdivide the given rectangle into four pieces: a $4a\times 4c$ rectangle, a $15\times(4c+0.5)$ fragment, a $(4a+0.5)\times 15$ fragment and the remaining corner fragment. This division is illustrated at the right-hand side of Figure 10. Note that in Figure 5, if we connect the smallest 15-width initial Fragment I to the (15,15)-Corner Fragment in the horizontal orientation, we get 27 full columns. Similarly in the vertical orientation, connecting a flipped Fragment I in Figure 5 to the (15,15)-Corner Fragment gives 27 full rows. Therefore, the smallest $(4j+3)\times(4k+3)$ rectangle that can be tiled using the scheme depicted in Figure 10 is a 27×27 rectangle.

The $4a\times 4c$ rectangle needs no monominoes by Theorem 1(i). The corner fragment will be tiled as illustrated in the tiling of the (15,15)-Corner Fragment in Figure 10. This needs three monominoes. Depending on the value of c, the $15\times(4c+0.5)$ fragment uses either initial fragments labeled I, K, M, or O in Figure 5 with possible extensions by 15-width T-tetromino cylinders. This would need only one monomino. Similarly, the $(4a+0.5)\times 15$ fragment would also need one monomino. In total, this tiling scheme of the whole rectangle needs only five monominoes, which is what we wanted to show. ■

In Figure 11, we present the optimal tilings of some rectangles that illustrate Lemmas 6, 7, and 8. We end this section with the statement of our main result.

Theorem 8: $G\leq 6$.

Proof: The result is immediate in view of Theorem 3, Lemma 6, Lemma 7, and Lemma 8. ■

CONCLUDING DISCUSSION

It remains an open problem whether $G=5$ or $G=6$. To show the former, we need to present a tiling scheme for all sufficiently large $(4j+1)\times(4k+2)$ and $(4j+2)\times(4k+3)$ rectangles that would only need two monominoes. We have been unable to construct an appropriate corner fragment that would do this job.

It is also possible that such tiling schemes do not exist and we get $G=6$. To prove this, we need to show that it is impossible to tile an infinite number of $(4j+1)\times(4k+2)$ or $(4j+2)\times(4k+3)$ rectangles using T-tetrominoes and only 2 monominoes. It has also struck us curious that many exceptions to the general conclusions of Theorems 6 and 7 in (Hochberg 2015) concern rectangles with at least one even side length. Maybe a non-tileability result such as in (Zhan 2012) could give a unified explanation for these exceptions.

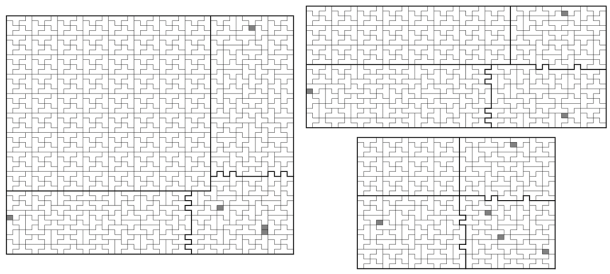


Figure 11: (Left) An optimal tiling of the $\mathbf{49\times 45}$ rectangle, illustrating Lemma 5. (Right above) An optimal tiling of the $\mathbf{25\times 47}$ rectangle, illustrating Lemma 6. (Right below) An optimal tiling of the $\mathbf{27\times 31}$ rectangle, illustrating Lemma 7.

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CONFLICT OF INTEREST

The authors declare that they do not have conflicts of interest.

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